Morse and Lyapunov Spectra and Dynamics on Flag Bundles*

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Abstract

This paper studies characteristic exponents of flows in relation with the dynamics of flows on flag bundles. The starting point is a flow on a principal bundle with semi-simple group G. Projection against the Iwasawa decomposition $G = KAN$ defines an additive cocycle over the flow with values in $\mathfrak{a} = \log A$. Its Lyapunov exponents (limits along trajectories) and Morse exponents (limits along chains) are studied. It is proved a symmetric property of these spectral sets, namely invariance under the Weyl group. It is proved also that these sets are located in certain Weyl chambers, defined from the dynamics on the associated flag bundles. As a special case linear flows on vector bundles are considered.

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1 Introduction

In this paper we study characteristic exponents (Morse and Lyapunov vector spectra) associated to continuous flows evolving on principal bundles. The purpose is to relate these exponents to the dynamics on the flag bundles. We work with invariant flows on a general principal bundle whose structure group G is semi-simple or reductive. The objects to be considered are defined intrinsically from G and the principal bundle. Thus our characteristic exponents take values on a vector subspace $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of G, while the flag bundles in question have fiber $\mathbb{F}$, a compact homeneous space of G (generalized flag manifolds of G). When G is a linear group $\mathfrak{a}$ becomes a subspace of diagonal matrices and $\mathbb{F}$ a manifold of flags of subspaces.

The vector valued spectra are defined via the cocycle over the flag bundle obtained by projection against an Iwasawa decomposition of G. They measure the exponential growth ratio of the flow the same way as the usual one-dimensional spectra measures the growth ratio of linear flows on vector bundles. (Actually the one-dimensional spectrum can be recovered from the vector valued one, via a representation of G, see Section 9.) This is the set up of Kaimanovich [16] to the proof of the multiplicative ergodic theorem of Oseledets.

The Morse spectrum set is a concept of growth ratio along chains introduced by Colonius-Kliemann [5], [6] (see also Colonius-Fabri-Johnson [7] for a vector valued version). Each one of these sets depend on a chain component of a flow. In our case the Morse spectra are defined over chain components of the flow on an associated flag bundle.

Our purpose here is to relate the geometry of the vector Lyapunov and Morse spectra with the geometry of the chain components on the flag bundle, which were studied in [4] and [22].

We better explain our results with the concrete example where $G = \mathrm{Sl}(3, \mathbb{R})$. Let X be a compact Hausdorff space endowed with a continuous flow $(t, x) \mapsto t \cdot x$ ($t \in \mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$) and suppose that $\phi_t(x, g) = (t \cdot x, \rho(t, x) g)$ is a flow on $Q = X \times G$ with ρ a continuous cocycle taking values in G. Let $A \subset G$ be the subgroup of diagonal matrices (positive entries) and N the subgroup of upper triangular unipotent matrices. Then the Iwasawa decomposition of G reads $G = KAN$ with $K = \mathrm{SO}(3)$. For $x \in X$ and $u \in K$ we write $\rho(t, x) u = k_t(x, u) a_t(x, u) n_t(x, u) \in KAN$ and look at the limit behavior as $t \to \infty$ of the matrix $\mathsf{a}(t, x, u)/t = \log a_t(x, u)/t$ in the subspace $\mathfrak{a} = \log A$.

This is done in combination with the asymptotics of the compact part $k_t(x,u)$. Via the action of G on $K \simeq G/AN$ the component $k_t(x,u)$ becomes a flow on K and $\mathsf{a}(t,x,k)$ an additive cocycle over this flow. Thus the characteristic exponents defined by a depend on $(x,k) \in X \times K$. Nevertheless it is more convenient to avoid ambiguities and factor out the subgroup $M \subset K$ of diagonal matrices with ± 1 entries and get $\mathsf{a}(t,x,b)$ as a cocycle over the flow $\phi^{\mathbb{F}}$ induced on $X \times \mathbb{F}$, where $\mathbb{F} \simeq K/M \simeq G/MAN$ is the manifold of complete flags in $\mathbb{R}^3$.

Now let $\mathcal{M}$ be a chain component of the flow $\phi^{\mathbb{F}}$ on $X \times \mathbb{F}$. Its associated Morse spectrum set $\Lambda_{\mathrm{Mo}}(\mathcal{M}) \subset \mathfrak{a}$ is defined by evaluating the cocycle $\mathsf{a}(t,x,b)$ on chains in $\mathcal{M}$ taking into account the jumps of the chains (see Definition 3.1 below). The Lyapunov spectrum set $\Lambda_{\mathrm{Ly}}(\mathcal{M})$ is the set of limits $\lim \mathsf{a}(t,x,b)/t$ along the trajectories in $\mathcal{M}$. By general facts $\Lambda_{\mathrm{Mo}}(\mathcal{M})$ is a compact convex set which contains $\Lambda_{\mathrm{Ly}}(\mathcal{M})$, which in turn contains the set of extremal points of $\Lambda_{\mathrm{Mo}}(\mathcal{M})$ (see Section 3 below).

The chain components in $X \times \mathbb{F}$ were described in [4], [22] with the assumption that the flow on X is chain transitive. The picture is the following: There exists $H_\phi \in \mathfrak{a}$ and a continuous map $x \in X \mapsto f_\phi(x) = g_x H_\phi g_x^{-1}$, $g_x \in G$, into the set of conjugates of H_ϕ such that each chain component $\mathcal{M}$ has the form $\bigcup_{x \in X} \{x\} \times \mathrm{fix}(f_\phi(x))$. Here $\mathrm{fix}(f_\phi(x))$ is a connected component of the fixed point set of the action on $\mathbb{F}$ of $\exp f_\phi(x)$ (the connected components can be labelled accordingly so that they match each other). For example if the diagonal matrix H_ϕ has distinct eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$ with eigenvectors e_1, e_2, e_3 then there are $3! = 6$ isolated fixed points, namely the flag $b_0 = (\mathrm{span}\{e_1\} \subset \mathrm{span}\{e_1, e_2\})$ together with those obtained from b_0 by permuting the basic vectors. A similar description holds for $f_\phi(x)$, $x \in X$, allowing to label the fixed points, as well as the chain components, with the permutation group on three letters S_3, which is the Weyl group $\mathcal{W}$ for $\mathrm{Sl}(3,\mathbb{R})$. In this case there are 6 distinct chain components denoted by $\mathcal{M}_w$ with w running over $\mathcal{W} = S_3$.

The chain components are also labelled by the permutation group even if H_ϕ has repeated eigenvalues, but now some of the $\mathcal{M}_w$ can merge. In our $\mathrm{Sl}(3,\mathbb{R})$ example there are four possibilities for H_ϕ, namely

(a) $H_\phi = \mathrm{diag}(\lambda_1 > \lambda_2 > \lambda_3)$ with six chain components $\mathcal{M}_w$, $w \in \mathcal{W}$.

(b) $H_\phi = \mathrm{diag}(\lambda_1 = \lambda_2 > \lambda_3)$ with three chain components $\mathcal{M}_1 = \mathcal{M}_{(12)}$, $\mathcal{M}_{(23)} = \mathcal{M}_{(123)}$ and $\mathcal{M}_{(132)} = \mathcal{M}_{(13)}$.

(c) $H_\phi = \mathrm{diag}(\lambda_1 > \lambda_2 = \lambda_3)$ with three chain components $\mathcal{M}_1 = \mathcal{M}_{(23)}$, $\mathcal{M}_{(12)} = \mathcal{M}_{(132)}$ and $\mathcal{M}_{(123)} = \mathcal{M}_{(13)}$.

(d) $H_\phi = \mathrm{diag}(\lambda_1 = \lambda_2 = \lambda_3) = 0$ and the flow is chain transitive on $X \times \mathbb{F}$.

(In all the cases $\mathcal{M}_1$ is the only attractor component and $\mathcal{M}_{(13)}$ the only repeller. The equalities in the second and third cases are due to the cosets $\mathcal{W}_\phi w$ where $\mathcal{W}_\phi$ is the subgroup of $\mathcal{W}$ fixing H_ϕ.)

We say that the matrix H_ϕ is the block form of ϕ. It is determined by the dynamics on the flag bundles and is a combination of the parabolic type of the flow and of the reversed flow (see [4], [22] and Section 5 of the present article).

The results of this paper provide a similar picture for the Morse and Lyapunov spectra over the chain components in terms of the Weyl group, the Weyl chambers and the block form H_ϕ of ϕ.

For $G = \mathrm{Sl}\,(3, \mathbb{R})$ the roots of $\mathfrak{a}$ are the functionals $\alpha_{ij}\,(\mathrm{diag}(a_1, a_2, a_3)) = a_i - a_j$, for $i \neq j \in \{1, 2, 3\}$. Their kernels are three straight lines containing the nonregular matrices in $\mathfrak{a}$, i.e., those having repeated eigenvalues. The complement to the lines is the union of $3! = 6$ Weyl chambers where the Weyl group acts simply and transitively (see Figure 1). A chamber is determined by inequalities $a_i > a_j > a_k$ and we denote it by $\mathcal{C}_w$ if $w \in \mathcal{W}$ is the permutation sending (123) to (ijk).

The main results of this paper prove the following estimates and symmetry properties of the vector valued Morse and Lyapunov spectra (see Figures 1 and 2):

1. The union $\bigcup_{w \in \mathcal{W}} \Lambda_{\mathrm{Mo}}\,(\mathcal{M}_w)$ of the several Morse spectra is invariant under the Weyl group. The same holds for the Lyapunov spectra $\bigcup_{w \in \mathcal{W}} \Lambda_{\mathrm{Ly}}\,(\mathcal{M}_w)$.

2. The spectra $\Lambda_{\mathrm{Mo}}\,(\mathcal{M}_1)$ and $\Lambda_{\mathrm{Ly}}\,(\mathcal{M}_1)$ of the attractor component are invariant under the subgroup $\mathcal{W}_\phi$ fixing H_ϕ.

3. $\Lambda_{\mathrm{Mo}}\,(\mathcal{M}_1)$ is contained in the interior of $\bigcup_{w \in \mathcal{W}_\phi} \mathrm{cl}\mathcal{C}_w$. This means that the multiplicities of the eigenvalues of any Morse exponent in $\Lambda_{\mathrm{Mo}}\,(\mathcal{M}_1)$ do not exceed those of H_ϕ. The same estimate holds for $\Lambda_{\mathrm{Ly}}\,(\mathcal{M}_1)$, since $\Lambda_{\mathrm{Mo}}\,(\mathcal{M}_w)$ is the convex closure of $\Lambda_{\mathrm{Ly}}\,(\mathcal{M}_w)$.

4. $\Lambda_{\mathrm{Mo}}\,(\mathcal{M}_1)$ intercepts the closure of every chamber $\mathcal{C}_w$, $w \in \mathcal{W}_\phi$. Hence there exists a Morse exponent in $\Lambda_{\mathrm{Mo}}\,(\mathcal{M}_1)$ whose eigenvalues have

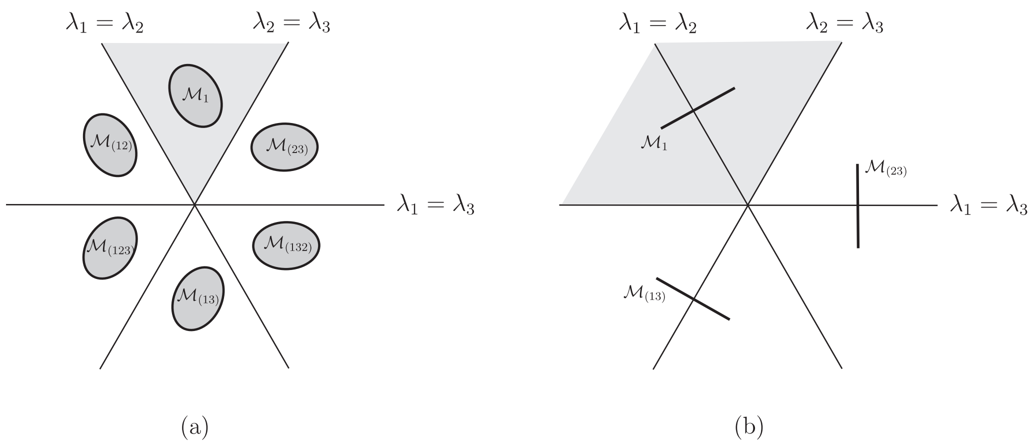

Figure 1: Morse spectra for flows with $G = \mathrm{Sl}(3, \mathbb{R})$: block forms (a) and (b), each spectrum is labeled by its corresponding component.

the same pattern of multiplicities as H_ϕ. $\Lambda_{\mathrm{Ly}}(\mathcal{M}_1)$ also meets these chambers, but since it is not necessarily convex, it may happen that every Lyapunov exponent has less multiplicities than H_ϕ.

5. $\Lambda_{\mathrm{Mo}}(\mathcal{M}_w) = w^{-1}\Lambda_{\mathrm{Mo}}(\mathcal{M}_1)$ and $\Lambda_{\mathrm{Ly}}(\mathcal{M}_w) = w^{-1}\Lambda_{\mathrm{Ly}}(\mathcal{M}_1)$. Combining this with the previous statement it follows that distinct Morse spectra do not overlap. (This fact is not true for linear flows on vector bundles as shown in [6], Example 5.5.11. The point here is that the vector bundle Morse spectra are images under linear maps of our spectra, see Section 9. Overlappings of the images may occur.)

The last statement says that the whole Morse spectra can be read off from the spectrum of the attractor component. This phenomenon is already present in the analysis of the chain components in flag bundles, whose main properties are governed by the extremal (attractor and repeller) components.

These results show an intimate relation between the spectra and the dynamics on the flag bundles. From the chain components on the bundles we get the symmetries of the spectra. Conversely, from the Morse spectra (of the attractor component only) we can recover the block form H_ϕ of the flow, and hence its parabolic type. This in turn tells the number of chain components, their geometry, their Conley indices, etc.

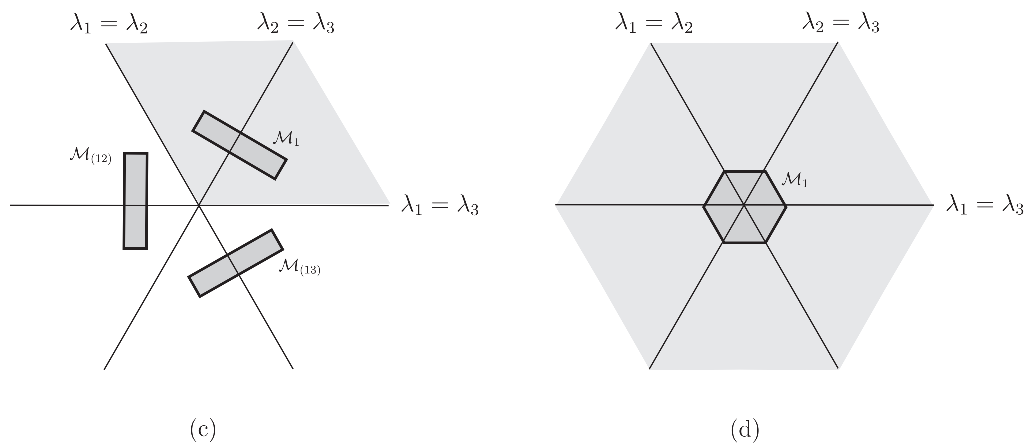

Figure 2: Morse spectra for flows with $G = \mathrm{Sl}(3, \mathbb{R})$: block forms (c) and (d).

The underlying structure behind this relationship is a "block decomposition" of the flow in the following sense: From the chain components on the flag bundles one can build a ϕ_t-invariant subbundle $Q_\phi \subset Q$ whose structure group is the centralizer Z_{H_ϕ} of the block form H_ϕ (see Section 5 below). If $G = \mathrm{Sl}\,(n, \mathbb{R})$ then Z_{H_ϕ} is a subgroup of block diagonal matrices, hence we say that Q_ϕ is the block reduction of ϕ. The above results say that the spectra have the same block structure as the block reduction, as expected. In case the bundles Q and Q_ϕ are trivial then the reduction amounts a cohomology of cocycles reducing the original cocycle to one taking values in the subgroup Z_{H_ϕ}. Hence this reduction points towards a Jordan decomposition of the flow, in the sense of Arnold-Cong-Oseledets [1].

A similar picture to the one described in the above examples holds for invariant flows on a general principal bundle whose structure group G is a noncompact semi-simple Lie group with finite center (the Morse spectra for the symplectic group $\mathrm{Sp}\,(4, \mathbb{R})$ are depicted in Figure 3 where $\mathfrak{a}$ consists of the diagonal matrices $\mathrm{diag}(\lambda_1, \lambda_2, -\lambda_1, -\lambda_2)$). We develop the setup and state our results in the generality of semi-simple Lie groups. These results can be extended to reductive groups (e.g. $\mathrm{Gl}\,(n, \mathbb{R})$). But now one must add a central component to $\mathfrak{a}$ and hence to Λ_{Mo}. There is not much to say about this central component unless that it is the same for every chain component $\mathcal{M}$.

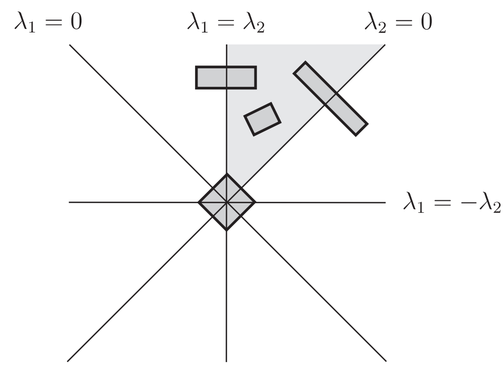

Figure 3: For flows with $G = \mathrm{Sp}(4, \mathbb{R})$ the spectrum lives here.

2 Preliminary notation

For the theory of semi-simple Lie groups and their flag manifolds we refer to Duistermat-Kolk-Varadarajan [11], Helgason [14] and Warner [30]. To set notation let G be a noncompact semi-simple Lie group with Lie algebra $\mathfrak{g}$. We assume throughout that G has finite center. Fix a Cartan involution θ of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. The form $B_\theta(X, Y) = -\langle X, \theta Y \rangle$, where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form of $\mathfrak{g}$, is an inner product.

Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. We let Π be the set of roots of $\mathfrak{a}$, Π^+ the positive roots corresponding to $\mathfrak{a}^+$ and Σ the set of simple roots in Π^+. The associated Weyl group is $\mathcal{W} = M^*/M$ where M^* and M are the normalizer and the centralizer of A, respectively. We write $\mathfrak{m}$ for the Lie algebra of M.

The Iwasawa decomposition reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space associated to α. Put $\mathfrak{n}^- = \sum_{\alpha \in -\Pi^+} \mathfrak{g}_\alpha$. As to the global decompositions we write $G = KS$ and $G = KAN$ with $K = \exp \mathfrak{k}$, $S = \exp \mathfrak{s}$, $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}^+$. The standard minimal parabolic subalgebra is defined by

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$$

Associated to a subset of simple roots $\Theta \subset \Sigma$ there are several Lie algebras and groups (cf. [30], Section 1.2.4): We write $\mathfrak{g}(\Theta)$ for the (semi-simple) Lie subalgebra generated by $\mathfrak{g}_\alpha$, $\alpha \in \Theta$, and put $\mathfrak{a}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{a}$ and $\mathfrak{n}^\pm(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{n}^\pm$. $G(\Theta)$ is the connected group with Lie algebra $\mathfrak{g}(\Theta)$ and $A(\Theta) = \exp \mathfrak{a}(\Theta)$, $N^\pm(\Theta) = \exp \mathfrak{n}^\pm(\Theta)$. Also, $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0,$

$\alpha \in \Theta\}$ and $A_\Theta = \exp \mathfrak{a}_\Theta$.

We let Z_Θ be the centralizer of $\mathfrak{a}_\Theta$ in G and $K_\Theta = Z_\Theta \cap K$. It decomposes as $Z_\Theta = MG(\Theta)A_\Theta$ which implies that $Z_\Theta = K_\Theta(S \cap Z_\Theta)$ is a Cartan decomposition while $Z_\Theta = K_\Theta AN(\Theta)$ is an Iwasawa decomposition of Z_Θ (which is a reductive Lie group).

The standard parabolic subagebra $\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p}$ and the corresponding standard parabolic subgroup P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G. The flag manifold of type Θ is $\mathbb{F}_\Theta = G/P_\Theta$, which identifies with the set of conjugates of $\mathfrak{p}_\Theta$. The empty set $\Theta = \emptyset$ gives the minimal parabolic subgroup $P = P_\emptyset$, which is $P = MAN$.

We write $\mathfrak{p}_\Theta^- = \theta(\mathfrak{p}_\Theta)$ for the parabolic subalgebra opposed to $\mathfrak{p}_\Theta$. It is conjugate to the parabolic subalgebra $\mathfrak{p}_{\Theta^*}$ where $\Theta^* = -(w_0)\Theta$ and w_0 is the principal involution of $\mathcal{W}$ (the one that takes Σ to $-\Sigma$). More precisely, $\mathfrak{p}_\Theta^- = \overline{w}_0\mathfrak{p}_{\Theta^*}$ if $\overline{w}_0 \in M^*$ is a representative of w_0. If P_Θ^- is the parabolic subgroup associated to $\mathfrak{p}_\Theta^-$ then

$$Z_\Theta = P_\Theta \cap P_\Theta^-,$$

and $P_\Theta^- = N_\Theta^- Z_\Theta$.

The subset Θ singles out the subgroup of the Weyl group $\mathcal{W}_\Theta = (\mathcal{W} \cap P_\Theta)/M = (\mathcal{W} \cap Z_\Theta)/M$. Alternatively $\mathcal{W}_\Theta$ is the subgroup generated by the reflections with respect to the roots $\alpha \in \Theta$.

For $H \in \mathfrak{a}$ let $\Theta_H = \{\alpha \in \Sigma : \alpha(H) = 0\}$. Then we denote the above subalgebras an groups with H instead of Θ_H, for example, $\mathfrak{p}_H = \mathfrak{p}_{\Theta_H}$, etc. (cf. Section 5). In this case $Z_H = Z_{\Theta_H}$ and $K_H = K_{\Theta_H}$ are the centralizer of H in G and K respectively. Also, $\mathcal{W}_H = \mathcal{W}_{\Theta_H}$ is the subgroup of $\mathcal{W}$ fixing H. Conversely, for appropriately chosen elements $H \in \mathfrak{a}$ such that $\Theta_H = \Theta$, the parabolic subgroups, flags, etc. can be written with reference to H.

3 Morse spectrum of a vector valued cocycle

In this section we discuss the abstract concepts of vector valued cocycles and their Lyapunov and Morse characteristic exponents. We mostly recall the results of Colonius-Kliemann [6] and Colonius-Fabbri-Johnson [7], which are stated in the continuous-time setting for flows on metric spaces. We indicate here the mild modifications needed for discrete-time flows on more general topological spaces. Detailed proofs will be given elsewhere [28].

Let E be a compact Hausdorff space and $\phi : \mathbb{T} \times E \to E$ a continuous flow with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. We denote $\phi(t, x) = \phi_t(x) = t \cdot x$. If V is a finite dimensional normed vector space, a V-valued cocycle over E is a continuous map $a : \mathbb{T} \times E \to V$ with

$$a(t + s, x) = a(t, s \cdot x) + a(s, x).$$

Two V-valued cocycles a and b over E are cohomologous with cohomology $h : E \to V$ if $a(t, x) + h(x) = h(t \cdot x) + b(t, x)$, where h is a continuous map. A V-valued cocycle $a(t, x)$ over E defines a flow on $E \times V$ by

$$t \cdot (x, v) = (t \cdot x,\ a(t, x) + v).$$

The product $E \times V$ can be viewed as principal V-bundle over E, where the aditive group V acts on the right and leaves invariant the above flow, which is thus made of automorphisms of the bundle. Conversely, let $\pi : P \to E$ be a principal bundle with vector structure group V and ϕ_t a flow of automorphisms of P. It is well known that P is a trivial bundle (see e.g. Kobayashi-Nomizu [18]). To each global continuous section $\chi : E \to P$ there corresponds a trivialization of P and a continuous V-cocycle $a_\chi(t, x)$ over E given by

$$t \cdot \chi(x) = \chi(t \cdot x) \cdot a_\chi(t, x).$$

We note that any two cocycles coming this way from sections are cohomologous. In fact, if $\eta : E \to P$ is another section of P and a_η the corresponding cocycle, then there exists a continuous map $h : X \to V$ satisfying $\chi(x) = \eta(x) \cdot h(x)$, and it follows that

$$a_\eta(t, x) + h(x) = h(t \cdot x) + a_\chi(t, x),$$

so that the map h realizes a cohomology between a_η and a_χ. Vector cocycles coming from such trivializations of principal bundles will show up below.

Given $x \in E$, $T \in \mathbb{T}$, the finite-time Lyapunov exponent of the cocycle a at (x, T) is

$$\lambda_T(x) = \frac{1}{T} a(T, x),$$

while the Lyapunov exponent of a at (in the direction of) x is

$$\lambda(x) = \lim_{T \to +\infty} \lambda_T(x),$$

if the limit exists. The Lyapunov spectrum of a subset $Y \subset E$ is defined by

$$\Lambda_{\mathrm{Ly}}(Y) = \{\lambda(y) \in V : \, y \in Y \,\mathrm{and}\, \lambda(y) \,\mathrm{exists}\}.$$

The exponential growth ratio along chains was introduced by Colonius-Kliemann [5], who called them Morse exponents. We use the approach to chains for flows in topological spaces as developed in Patrão [20]. Thus let $\mathcal{O}$ be a family of open coverings of the compact Hausdorff space E, which is admissible in the sense of [20]. If $\mathcal{U} \in \mathcal{O}$ is given then two points $x, y \in E$ are said to $\mathcal{U}$-close if there exists an open set $A \in \mathcal{U}$ such that $x, y \in A$.

A $(\mathcal{U}, T)$-chain ($\mathcal{U} \in \mathcal{O}$, $T \in \mathbb{T}$) ζ from x to y in E is a finite sequence of points $x = x_0, x_1, \ldots, x_N$ in E and times $t_0, t_1, \ldots, t_{N-1}$ in $\mathbb{T}$ with $t_i \geq T$ such that $t_i \cdot x_i$ and x_{i+1} are $\mathcal{U}$-close for $i = 0, \ldots, N-1$. We say that the chain ζ belongs to the subset $\mathcal{M} \subset E$ if the initial and end points x, y belong to $\mathcal{M}$ (but not necessarily the intermediate points).

Let a as above be a cocycle over E.

Definition 3.1. *The finite time Morse exponent $\lambda(\zeta)$ of the $(\mathcal{U}, T)$-chain ζ is*

$$\lambda(\zeta) = \frac{1}{T(\zeta)} \sum_{j=0}^{N-1} a(T_j, x_j),$$

where $T(\zeta) = \sum_{j=0}^{N-1} T_j$ is the total time of ζ. We denote by $\Lambda_{\mathrm{Mo}}(\mathcal{M}; \mathcal{U}, T)$ the set of $(\mathcal{U}, T)$-chains belonging to $\mathcal{M}$ and define the Morse spectrum of $\mathcal{M}$ to be

$$\Lambda_{\mathrm{Mo}}(\mathcal{M}) = \bigcap \{\mathrm{cl}\Lambda_{\mathrm{Mo}}(\mathcal{M}; \mathcal{U}, T) : \mathcal{U} \in \mathcal{O}, \, T > 0\}.$$

Note that the family of coverings $\mathcal{O}$ is an ingredient in the definition of $\Lambda_{\mathrm{Mo}}(\mathcal{M})$. However this set does not depend on the specific choice of $\mathcal{O}$ (see Corollary 3.4 below).

It follows immediately from the definitions that

$$\lambda(\zeta) = \sum_{j=0}^{N-1} \frac{T_j}{T(\zeta)} \lambda_{T_j}(x_j),$$

so that each finite-time Morse exponent is a convex combination of finite-time Lyapunov exponents. That an analogous result is valid for the limit Morse exponents is one of the central results on the structure of Morse exponents which is stated next, together with other results.

Theorem 3.2. *Let $\mathcal{M} \subset E$ be a chain component (that is, a maximal chain transitive set).*

1. *Two cohomologous cocycles have the same Lyapunov and Morse spectra.*

2. *The Morse spectrum $\Lambda_{\mathrm{Mo}}(\mathcal{M})$ of a continuous time flow coincides with any of its discretezations (restriction to $c\mathbb{Z}$, $c \in \mathbb{R}$).*

3. *Suppose that $\mathcal{M}$ contains the ω-limit set $\omega(x)$ of $x \in E$. Then $\lambda(x) \in \Lambda_{\mathrm{Mo}}(\mathcal{M})$, if $\lambda(x)$ exists.*

4. *$\Lambda_{\mathrm{Mo}}(\mathcal{M})$ is a nonempty compact convex subset of V.*

5. *$\Lambda_{\mathrm{Mo}}(\mathcal{M})$ is the set of integrals $\int q \, \mathrm{d}\mathbb{P}$ with $\mathbb{P}$ running through the probability measures supported in $\mathcal{M}$ which are invariant by the time-one flow ϕ_1. Here $q : E \to V$ is the time-one map $q(x) = a(1, x)$.*

6. *The extremal points of $\Lambda_{\mathrm{Mo}}(\mathcal{M})$ are Lyapunov exponents so that $\Lambda_{\mathrm{Mo}}(\mathcal{M})$ is the closed convex closure of $\Lambda_{\mathrm{Ly}}(\mathcal{M})$.*

Proof: See [7]. For item (2) and proofs in the present more general context see [28]. □

Some immediate applications of these results are given in what follows.

Corollary 3.3. *Let $\mathcal{M}_j$, $j = 1, \ldots, n$, be the finest Morse decomposition of ϕ_t in E. Then*

$$\Lambda_{\mathrm{Ly}}(E) \subset \bigcup_{j=1}^{n} \Lambda_{\mathrm{Mo}}(\mathcal{M}_j).$$

Proof: In fact, the Morse components $\mathcal{M}_j$ are chain components and contain all the ω-limit sets in E. The result then follows from the above theorem. □

Corollary 3.4. *The Morse spectrum of a chain component $\mathcal{M}$ does not depend on the family of coverings $\mathcal{O}$.*

Proof: In fact, by the above theorem we have $\Lambda_{\mathrm{Mo}}(\mathcal{M}) = \mathrm{cl}\,(\mathrm{co}\Lambda_{\mathrm{Ly}}(\mathcal{M}))$. Since $\Lambda_{\mathrm{Ly}}(\mathcal{M})$ does not depend on $\mathcal{O}$ the result follows. □

Corollary 3.5. *Let ϕ'_t be a flow on a compact Hausdorff space E' and let b be a V'-vector cocycle over E'. Suppose that there is a map $\pi : E \to E'$ and a linear map $p : V \to V'$ such that the cocycles a and b are related by*

$$b(t, \pi(x)) = p(a(t, x)), \quad t \in \mathbb{T},\ x \in E,$$

and that $\pi(\mathcal{M})$ is a chain transitive component in E'. Then

$$\Lambda_{\mathrm{Mo}}(\pi(\mathcal{M}), b) = p(\Lambda_{\mathrm{Mo}}(\mathcal{M}, a)), \tag{1}$$

where the notation indicates which cocycle is being considered.

Proof: The Lyapunov exponents of a and b are easily seen to be related by (1) with Λ_{Ly} in place of Λ_{Mo}. The result then follows by the above theorem. $\square$

4 Decompositions of principal bundles and the $\mathfrak{a}$-cocycle

Let $Q \to X$ be a principal bundle whose structural group G is a reductive Lie group. In this section we build decompositions of Q similar to the Iwasawa and Cartan decompositions of G. We assume throughout that the base space X is paracompact. The right action of G on Q is denoted by $q \mapsto q \cdot g$, $q \in Q$, $g \in G$.

4.1 Iwasawa decomposition

An Iwasawa decomposition $G = KAN$ of the structural group can be carried over to a decomposition of the bundle, which we call the *Iwasawa decomposition* of Q as well. The construction goes as follows. The group AN is diffeomorphic to an Euclidian space and since the base space X of Q is paracompact, by general results on principal bundles there exists a K-reduction of Q, that is, a subbundle $R \subset Q$ which is a principal bundle with structural group K (for a proof combine Theorem I.5.7 with Proposition I.5.6 in Kobayashi-Nomizu [18]). In case G is a linear group such reduction can be obtained from a Riemannian metric in an associated vector bundle.

For every $q \in Q$ there exists $g \in G$ and $r' \in R$ such that $q = r' \cdot g$. The K-component of the Iwasawa decomposition $g = khn \in KAN$ of g leaves

R invariant so that, by uniqueness of the Iwasawa decomposition, it follows that every $q \in Q$ decomposes uniquely as

$$q = r \cdot hn, \quad r \in R, \quad hn \in AN,$$

which exhibits the bundle Q as a product $Q = R \cdot AN \approx R \times A \times N$. We let

$$\mathsf{R} : Q \to R, \quad q \mapsto r, \qquad \mathsf{A} : Q \to A, \quad q \mapsto h,$$

be the associated projections. By the continuity of the Iwasawa decomposition of G and the local triviality of Q, it follows that these projections are continuous. Moreover they satisfy the following properties:

1. $\mathsf{R}(r) = r$, $\mathsf{A}(r) = 1$, when $r \in R$,
2. $\mathsf{R}(q \cdot p) = \mathsf{R}(q)m$, $\mathsf{A}(q \cdot p) = \mathsf{A}(q)h$, when $q \in Q$, $p = mhn \in P = MAN$. In particular, $\mathsf{A}(r \cdot p) = h$.

In what follows we write for $q \in Q$,

$$\mathsf{a}(q) = \log \mathsf{A}(q) \in \mathfrak{a}.$$

We use the same notation for the Iwasawa decomposition of $g \in G$, namely, $\mathsf{a}(g) = \log h$ if $g = uhn \in KAN$. For future reference we note that by the second of the above properties we have

$$\mathsf{a}(q \cdot p) = \mathsf{a}(q) + \mathsf{a}(p), \quad p \in P. \tag{2}$$

Now we discuss the cocycle defined by a. The first step is to get actions on the K-subbundle R. Given an automorphism $\varphi \in \mathrm{Aut}(Q)$ define the map

$$\varphi^R : r \in R \mapsto \mathsf{R}(\varphi(r)) \in R.$$

We have

$$\varphi^R \circ \psi^R = (\varphi \circ \psi)^R$$

if $\psi \in \mathrm{Aut}(Q)$. In fact, if $\psi(r) = r_1 \cdot g$, $r_1 \in R$, $g \in AN$, then $\psi^R(r) = r_1$ so that by the above properties

$$(\varphi \circ \psi)^R(r) = \mathsf{R}(\varphi(\psi(r))) = \mathsf{R}(\varphi(r_1) \cdot g) = \mathsf{R}(\varphi(r_1)) = \varphi^R(r_1) = \varphi^R(\psi^R(r)).$$

We remark that φ^R is not a bundle morphism unless R is φ-invariant. Also, when $Q = G$ then φ^R reduces to the usual action of G on K through the Iwasawa decomposition of G.

Let ϕ_t be the continuous flow of automorphisms of Q, $t \in \mathbb{T}$. Then ϕ_t^R defines a continuous flow in R. We use also the notation a for the cocycle defined by the map a, namely

$$\mathsf{a} : \mathbb{T} \times R \to \mathfrak{a}, \quad \mathsf{a}(t, r) = \mathsf{a}(\phi_t^R(r)),$$

which is continuous by the continuity of ϕ and the projection a. It follows from the properties of the Iwasawa decomposition of Q that a is a cocycle over ϕ_s^R, that is,

$$\mathsf{a}(t + s, r) = \mathsf{a}(t, \phi_s^R(r)) + \mathsf{a}(\phi_s, r),$$

where $t, s \in \mathbb{T}$, $r \in R$.

In what follows we write simply ϕ_t instead of ϕ_t^R.

4.2 $\mathfrak{a}$-cocycle over flag bundles

The cocycle a over R can be factored to a cocycle over the flag bundle $\mathbb{F}Q$, which is the associated bundle $Q \times_G \mathbb{F}$ obtained by the left action of G on $\mathbb{F}$. The construction is as follows:

Take the closed subgroup MN of G. The quotient space Q/MN can be identified with the associated bundle $Q \times_G G/MN$ (see [18], Proposition 5.5). Let $P = MAN$ be the minimal parabolic subgroup, so that the flag bundle $\mathbb{F}Q \simeq Q/P$. Then there is a natural fibration $Q/MN \to Q/P = \mathbb{F}Q$, $q \cdot MN \mapsto q \cdot P$, $q \in Q$, whose fiber is $P/MN \simeq A$. Since MN is normal in P, it follows that this fibration is a principal bundle over $\mathbb{F}Q$ with structural group A. This bundle is trivializable (because A is diffeomorphic to an Euclidian space, see [18]). An explicit global cross section is given by a K-reduction R of Q. In fact, any element of $\mathbb{F}Q$ can be written $r \cdot b_0$, $r \in R$, where b_0 is the origin of $\mathbb{F}$. Consider the map

$$\chi(r \cdot b_0) = r \cdot MN \in Q/MN.$$

It is well defined because $r_1 \cdot b_0 = r_2 \cdot b_0$ implies that $r_2 = r_2 \cdot m$, $m \in M$, and hence $r_1 \cdot MN = r_2 \cdot MN$. It is clearly a cross section of $Q/MN \to \mathbb{F}Q$. Its continuity follows from the equality $\chi(q \cdot b_0) = \mathsf{R}(q) \cdot MN$.

Now if ϕ_t is a flow of automorphisms of Q, $t \in \mathbb{T}$, then the cross section χ defines a continuous $\mathfrak{a}$-valued cocycle $\mathsf{a} : \mathbb{T} \times \mathbb{F}Q \to \mathfrak{a}$ by $\mathsf{a}(t, \xi) = \log a_t$ where

$$\phi_t(\chi(\xi)) = \chi(\phi_t(\xi)) \cdot a_t, \quad \xi \in \mathbb{F}Q$$

(cf. Section 3). This is essentially the cocycle over R defined from the Iwasawa decomposition (which justifies our use of the same notation). In fact, let $\xi = r \cdot b_0 \in \mathbb{F}Q$ where $r \in R$ and b_0 is the origin of $\mathbb{F}$. Take the Iwasawa decomposition $\phi_t(r) = r_t \cdot a_t n_t \in R \cdot AN$. Then $\phi_t(\xi) = r_t b_0$ so that $\chi(\phi_t(\xi)) = r_t \cdot MN$. Also, since $n_t \in N$ fixes MN, we have $\phi_t(\chi(\xi)) = r_t \cdot a_t MN = \chi(\phi_t(\xi)) \cdot a_t$, in terms of the right action of A on Q/MN. This means that

$$\mathsf{a}(t, r \cdot b_0) = \log a_t = \mathsf{a}(t, r).$$

4.3 Cartan and polar decompositions

Fix a Cartan decomposition $G = KS$ with K as in the Iwasawa decomposition and the corresponding decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ of the Lie algebra $\mathfrak{g}$ of G. As before let R be a K-reduction of Q. Then for every $q \in Q$ there exists $g \in G$ and $r' \in R$ such that $q = r' \cdot g$. The K-component of the Cartan decomposition $g = ks$ of g leaves R invariant so that, by the uniqueness of the Cartan decomposition in G, it follows that every $q \in Q$ can be written uniquely as

$$q = r \cdot s, \quad r \in R, \quad s \in S$$

which exhibits the bundle Q as the product $Q \approx R \times S$. We let

$$\mathsf{R}^C : Q \to R, \quad q \mapsto r, \qquad \mathsf{S} : Q \to S, \quad q \mapsto s$$

be the associated projections. (We emphasize that the two projections R and R^C against the Iwasawa and Cartan decompositions, respectively, are not equal.) By the continuity of the Cartan decomposition of G and the local triviality of Q, it follows that these projections are continuous. Moreover they satisfy the following properties:

1. $\mathsf{R}^C(q \cdot k) = \mathsf{R}^C(q) \cdot k$ and $\mathsf{S}(q \cdot k) = k^{-1}\mathsf{S}(q)k$ if $q \in Q$ and $k \in K$. (This is a consequence of the fact that $kSk^{-1} = S$.)

2. $\mathsf{R}^C(r) = r$ and $\mathsf{S}(r \cdot g) = \mathsf{S}(g)$ if $r \in R$, $g \in G$, where we denote also by $\mathsf{S}(g)$ the S-component of $g \in G = K \times S$.

3. $\mathsf{S}(q \cdot g) = \mathsf{S}(\mathsf{S}(q)g)$ if $q \in Q$ and $g \in G$. (In fact, $q \cdot g = \mathsf{R}(q) \cdot \mathsf{S}(q)g$ so that if $\mathsf{S}(q)g = kt$ is the Cartan decomposition of $\mathsf{S}(q)g$ in G where $t = \mathsf{S}(\mathsf{S}(q)g)$ then $q \cdot g = \mathsf{R}(q) \cdot kt$. Hence $\mathsf{S}(q \cdot g) = t = \mathsf{S}(\mathsf{S}(q)g)$.)

Now fix a Weyl chamber A^+ sitting inside a maximal abelian $A \subset S$ and consider the polar decomposition $G = K(\mathrm{cl}A^+)K$. If $g = ks \in KS$ then $g = uhv$ where $u = kv^{-1} \in K$ and $s = v^{-1}hv$, $h \in \mathrm{cl}A^+$. Combining this polar decomposition with the Cartan decomposition of Q we can write every $q \in Q$ as

$$q = r \cdot hv, \quad r \in R, \quad h \in \mathrm{cl}A^+, \quad v \in K.$$

Here the component $h \in \mathrm{cl}A^+$ is uniquely defined (although r and v are not). Thus we have a well defined map

$$\mathsf{A}^+ : Q \to \mathrm{cl}A^+, \quad q \mapsto h$$

which satisfies $\mathsf{A}^+(q \cdot k) = \mathsf{A}^+(q)$ if $q \in Q$, $k \in K$.

In the sequel we denote with the corresponding lower case letters the logarithms of the above maps:

$$\mathsf{s}(q) = \log \mathsf{S}(q) \in \mathfrak{s} \qquad \mathsf{a}^+(q) = \log \mathsf{A}^+(q) \in \mathrm{cl}\mathfrak{a}^+.$$

If one starts instead with a Z_Θ-principal bundle Q_Θ, using the Cartan and Iwasawa decompositions of Z_Θ and the same arguments as above one gets the existence of a K_Θ-reduction R_Θ of Q_Θ, a Cartan decomposition of $q \in Q_\Theta$ given by

$$q = rs, \quad r \in R_\Theta, \quad s \in S \cap Z_\Theta,$$

and an Iwasawa decomposition of $q \in Q_\Theta$ given by

$$q = rhn, \quad r \in R_\Theta, \quad h \in A, n \in N(\Theta).$$

Also, if Q_Θ is a Z_Θ-reduction of the G-principal Q, then one can choose a K-reduction R of Q so that $R_\Theta \subset R$.

4.4 Reductive groups

The above cocycles can be also be defined for flows evolving on principal bundles with reductive Lie groups. This is because the Lie algebra $\overline{\mathfrak{g}}$ of a reductive Lie group $\overline{G}$ decomposes as $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{z}$, with $\mathfrak{g}$ semi-simple and

$\mathfrak{z}$ the center. Hence one can add a central component to $\mathfrak{a}$ (that is, the split part of $\mathfrak{z}$) and get a cocycle on a larger vector space. Also, under natural conditions there is available Cartan and Iwasawa decompositions of $\overline{G}$ as well as parabolic subgroups and flag manifolds, allowing analogous decompositions of $\overline{G}$-principal bundles. We refer to Knapp [17], Chapter VII, for the theory of reductive Lie groups.

We will not pursue here the detailed extension of the decompositions for reductive principal bundles. Instead we draw some comments regarding the group $\mathrm{Gl}\,(n,\mathbb{R})$, which show up in the vector bundles (see Section 9). This group is reductive with Iwasawa decomposition $\mathrm{O}(n,\mathbb{R})\,\overline{A}N$, where N is the unipotent upper triangular group and $\overline{A}$ the group of diagonal matrices with positive entries. The Lie algebra $\overline{\mathfrak{a}}$ of $\overline{A}$ decomposes as $\overline{\mathfrak{a}} = \mathbb{R}\cdot \mathrm{id} \oplus \mathfrak{a}$ where $\mathfrak{a}$ is the subspace of zero trace diagonal matrices. Thus the $\overline{\mathfrak{a}}$-valued cocycle $\overline{\mathsf{a}}(t,\xi)$ (constructed the same way as above) writes $\overline{\mathsf{a}}(t,\xi) = \mathsf{a}_0(t,\xi)\mathrm{id} + \mathsf{a}(t,\xi)$, $\xi \in \mathbb{F}Q$, where $\mathsf{a}_0(t,\xi)$ is a real valued cocycle which is essentially a trace of a matrix. Actually a_0 is constant as a function of ξ because scalar matrices belong to the center of G. Finally the action of $\mathrm{Gl}\,(n,\mathbb{R})$ factors to an action of $\mathrm{Sl}\,(n,\mathbb{R})$ (see [4]), so the dynamics on the flag bundle is the same as for a flow on a $\mathrm{Sl}(n,\mathbb{R})$-bundle.

4.5 Spectra

In the sequel we denote by $\Lambda_{\mathrm{Mo}}(\mathcal{M})$ and $\Lambda_{\mathrm{Ly}}(\mathcal{M})$ the Morse and Lyapunov vector spectra for the $\mathfrak{a}$-cocycle over the flag bundle $\mathbb{F}Q$ defined via the trivializable principal bundle $Q/MN \to \mathbb{F}Q$. In these expressions $\mathcal{M}$ is a chain component of the flow on $\mathbb{F}Q$.

For latter reference we note that these spectra does not depend on the trivialization of $Q/MN \to \mathbb{F}Q$ (and of the K-reduction of Q). This is because two different trivializations yield cohomologous cocycles, which in turn have the same spectra, since the base is compact.

5 Dynamics on flag bundles and block reduction

In this section we describe a ϕ_t-invariant subbundle $Q_\phi \subset Q \to X$ whose structure group is the centralizer Z_{H_ϕ} of an element $H_\phi \in \mathfrak{a}$ (or of its exponential $h_\phi = \exp H_\phi \in A$). In case G is a linear group H_ϕ is a diagonal

matrix and the centralizer Z_{H_ϕ} is a subgroup of block diagonal matrices (if the real eigenvalues of H_ϕ are ordered decreasingly). By analogy to this case we call Q_ϕ the block reduction and H_ϕ the block form of ϕ_t.

The invariant subbundle Q_ϕ is constructed after the results of [4], [22] that give the chain recurrent components of ϕ_t on the flag bundle $\mathbb{F}Q$. These chain components in turn are modelled fiberwise by the dynamics of the flow on the flag manifold $\mathbb{F}$ defined by a one-parameter subgroup $\exp tY$ of G. The description of this dynamics can be found [11], Section 3. In the next paragraphs we recall its main features and establish related notation. For the details and proofs see [11].

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition and $\mathfrak{a} \subset \mathfrak{s}$ a maximal abelian subspace. Then for every $H \in \mathfrak{a}$ the vector field $\widetilde{H}$ induced on $\mathbb{F}$ has flow $\exp tH$ and is the gradient of a function with respect to a Riemannian metric (Borel metric) in $\mathbb{F}$. Let $Z_H = \{g \in G : \mathrm{Ad}\,(g)\, H = H\}$ be centralizer of H in G and put $K_H = Z_H \cap K$. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be a Weyl chamber and suppose that $H \in \mathrm{cl}\mathfrak{a}^+$. If b_0 denotes the origin of $\mathbb{F}$ (corresponding to $\mathfrak{a}^+$) then the connected components of the singularity set of $\widetilde{H}$ are the orbits $K_H \cdot wb_0$ with w running through the Weyl group $\mathcal{W}$, which coincide with $Z_H \cdot wb_0$. These singularities are the fixed points the one-parameter subgroup $\exp(tH)$, $t \in \mathbb{R}$. These singularities are transversally hyperbolic submanifolds of $\mathbb{F}$ (that is, $\widetilde{H}$ has Morse-Bott dynamics) where $K_H \cdot b_0$ is the only attractor and $K_H \cdot w_0 b_0$ is the only repeller, where w_0 is the element of $\mathcal{W}$ with largest length (principal involution). Also, $K_H \cdot w_1 b_0 = K_H \cdot w_2 b_0$ if and only if $w_1 w_2^{-1} H = H$, that is, if and only if the cosets $\mathcal{W}_H w_1$ and $\mathcal{W}_H w_2$ are equal, where $\mathcal{W}_H$ is the subgroup of $\mathcal{W}$ fixing H. In particular, if $H \in \mathfrak{a}^+$ is regular then there are $|\mathcal{W}|$ isolated singularities.

The stable (respectively unstable) manifold of the component $K_H \cdot wb_0$ is the orbit $P_H^- \cdot wb_0 = N_H^- K_H \cdot wb_0$ (respectively $P_H^+ \cdot wb_0 = N_H^+ K_H \cdot wb_0$) of the parabolic subgroup P_H^- (respectively P_H^+) which is the normalizer of the parabolic subalgebra $\mathfrak{p}_H^-$ defined as the sum of the eigenspaces of $\mathrm{ad}\,(H)$ associated with the eigenvalues ≤ 0 (respectively ≥ 0).

If $Y = \mathrm{Ad}\,(g)\, H$, $H \in \mathrm{cl}\mathfrak{a}^+$, then g conjugates the induced vector fields $\widetilde{H}$ and $\widetilde{Y}$, so that the singularities of $\widetilde{Y}$ are $g\,(K_H \cdot wb_0)$ with $g\,(K_H \cdot b_0)$ the attractor and $g\,(K_H \cdot w_0 b_0)$ the repeller. Their stable and unstable manifolds are respectively $g\left(P_H^{\mp} \cdot wb_0\right)$. In the sequel we write $\mathrm{fix}\,(Y, w)$ for $g\,(K_H \cdot wb_0)$, the w-fixed points of Y, and $\mathrm{st}\,(Y, w)$ for the corresponding stable manifold $g\left(P_H^- \cdot wb_0\right)$.

Example: If $G = \mathrm{Sl}(n, \mathbb{R})$ then $\mathbb{F}$ is the manifold of complete flags $(V_1 \subset \cdots \subset V_{n-1})$ of subspaces $V_i \subset \mathbb{R}^n$ with $\dim V_i = i$. We can take $\mathfrak{a}$ to be the space zero trace of diagonal matrices and $\mathfrak{a}^+$ the cone of the matrices whose eigenvalues are ordered decreasingly. If $H \in \mathfrak{a}^+$ then $\widetilde{H}$ has isolated singularities which are the flags spanned by eigenvalues of H. The attractor is $b_0 = (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots)$, and if $w \in \mathcal{W}$ is a permutation then $\mathrm{fix}(H, w)$ is obtained by w-permuting the basic vectors from b_0. A nonregular $H \in \mathrm{cl}\mathfrak{a}^+$ has repeated eigenvalues which appear in blocks, and Z_H is the subgroup of block diagonal matrices, where the blocks have the same size as those of H. K_H is the subgroup of orthogonal matrices in Z_H. Then it is easy to write down the fixed point components $K_H \cdot wb_0$. For instance the attractor $\mathrm{fix}(H, 1)$ is made of flags adapted to the block decomposition of H.

Let $\pi : Q \to X$ be a principal bundle with semi-simple structural group G and compact Hausdorff base space X. Let ϕ_t be a right invariant flow on Q which is chain transitive on X. The flow ϕ_t induces a flow in the flag bundle $\mathbb{F}Q$. The following theorem of [4], [22] gives the chain components of this induced flow.

Theorem 5.1. *The flow ϕ_t on the maximal flag bundle $\mathbb{F}Q \to X$ admits a finest Morse decomposition, whose Morse components are determined as follows: There exists $H_\phi \in \mathrm{cl}\mathfrak{a}^+$ and a ϕ_t-invariant map*

$$f_\phi : Q \to \mathrm{Ad}(G)H_\phi \qquad f_\phi(\phi_t(q)) = f_\phi(q)$$

into the adjoint orbit of H_ϕ, which is equivariant, that is, $f_\phi(q \cdot g) = \mathrm{Ad}(g^{-1}) f_\phi(q)$, $q \in Q$, $g \in G$. The chain components $\mathcal{M}(w)$ are parameterized by the Weyl group $\mathcal{W}$ and each $\mathcal{M}(w)$ is given fiberwise as the fixed point set

$$\mathcal{M}(w)_{\pi(q)} = q \cdot \mathrm{fix}(f_\phi(q), w), \quad q \in Q.$$

There is just one attractor component $\mathcal{M}^+ = \mathcal{M}(1)$ and only one repeller $\mathcal{M}^- = \mathcal{M}(w_0)$ and their dynamical ordering is the reverse of the algebraic Bruhat-Chevalley order of $\mathcal{W}$.

In this theorem the case $H_\phi = 0$ is not ruled out. In this case the flow is chain transitive on the flag bundles.

In a partial flag bundle $\mathbb{F}_\Theta Q$, $\Theta \subset \Sigma$, there exists also the finest Morse decomposition, whose components $\mathcal{M}_\Theta(w)$ are the projections of the components $\mathcal{M}(w) \subset \mathbb{F}Q$. These projections are also given fiberwise as fixed points of $f_\phi(q)$, $q \in Q$.

Let $\Theta_\phi = \{\alpha \in \Sigma : \alpha(H_\phi) = 0\}$. Then the corresponding flag bundle $\mathbb{F}_{\Theta_\phi}Q$ has special properties. Namely, the attractor chain component $\mathcal{M}^+_{\Theta_\phi} = \mathcal{M}_{\Theta_\phi}(1)$ meets each fiber in a singleton, because the attractor fixed point component of H_ϕ on $\mathbb{F}_{\Theta_\phi}$ reduces to a point. The flag $\mathbb{F}_{\Theta_\phi}$ (or the corresponding parabolic subgroup P_{Θ_ϕ}, or else Θ_ϕ) were called the parabolic type of ϕ in [4], [22]. The parabolic type of the reversed flow is the dual Θ^*_ϕ (see [4]) and in the flag bundle $\mathbb{F}_{\Theta^*_\phi}Q$ the repeller component $\mathcal{M}^-_{\Theta^*_\phi}$ meets the fibers in singletons. Furthermore the centralizer $Z_{H_\phi} = P_{\Theta_\phi} \cap P^-_{\Theta_\phi}$. In the sequel we say that H_ϕ is the block form of ϕ.

Now let $Q_\phi = f_\phi^{-1}(H_\phi) \subset Q$. Since the adjoint orbit $\mathrm{Ad}(G)\, H_\phi$ is identified to the homogeneous space G/Z_{H_ϕ}, it follow by a well known fact that Q_ϕ is a subbundle of Q with structural group Z_{H_ϕ} (see Kobayashi-Nomizu [18], Proposition I.5.6, where the equivariant map f_ϕ must be viewed as a section of the associated bundle $Q \times_G G/Z_{H_\phi}$). The ϕ_t-invariance of f_ϕ implies the invariance of Q_ϕ under the flow.

Definition 5.2. *If H_ϕ is the block form of the flow then the Z_{H_ϕ}-subbundle Q_ϕ is called the block reduction of ϕ.*

Remark: If $Q \simeq X \times G$ is a trivial bundle then the flow is defined by a cocycle $\rho(t,x)$ with values in G: $\phi_t(x,g) = (t \cdot x, \rho(t,x)\, g)$. By choosing another trivialization of Q the cocycle is changed by cohomologous one. Now the block reduction Q_ϕ is a subbundle of a trivial bundle. It may not be trivial. But if this happens then the original cocycle is cohomologous to a cocycle taking values in the subgroup Z_{H_ϕ}.

Remark: The subgroup Z_{H_ϕ} is algebraic, hence it contains the algebraic hull of the flow (see Feres [12], Zimmer [31]).

The centralizer Z_{H_ϕ} is a reductive Lie subgroup of G. As such it has an Iwasawa $Z_{H_\phi} = K_{H_\phi} A N_{H_\phi}$, with compact component K_{H_ϕ} (see Knapp [17], Chapter VII). Hence, as observed in Section 4.3, the Z_{H_ϕ}-bundle admits a reduction to the compact group K_{H_ϕ}. We denote this subbundle by R_ϕ.

The following statement expresses the chain components on the flag bundles in terms of the block reduction.

Proposition 5.3. *For $\Theta \subset \Sigma$, let $\mathcal{M}_\Theta(w)$, $w \in \mathcal{W}$, be a chain component of ϕ_t on the flag bundle $\mathbb{F}_\Theta Q$. Then*

$$\mathcal{M}_\Theta(w) = \{q \cdot wb_\Theta : q \in Q_\phi\} = \{r \cdot wb_\Theta : r \in R_\phi\},$$

where b_Θ is the origin of $\mathbb{F}_\Theta$.

Proof: In fact, if $q \in Q_\phi$ then $\mathcal{M}_\Theta(w)_{\pi(q)} = q \cdot \mathrm{fix}(H_\phi, w)$, but $\mathrm{fix}(H_\phi, w) = Z_{H_\phi} w b_\Theta$, because $H_\phi \in \mathrm{cl}\mathfrak{a}^+$. Then the first equality follows since Z_{H_ϕ} acts transitively on the right in the fiber of q in Q_ϕ. The proof of the second equality is similar, by taking K_{H_ϕ} and R_ϕ instead of Z_{H_ϕ} and Q_ϕ. $\square$

We conclude this section with the construction of a linearization around the attractor of the induced flow in the flag bundle $\mathbb{F}_{\Theta(\phi)}Q$. This linearization is a flow which evolves on a subbundle of the tangent bundle to the fibers of $\mathbb{F}_\Theta Q$. It will be the main technical tool in the proof of the estimates of Section 7. More details about this construction can be found in the forthcoming paper [23].

We write $\Theta = \Theta(\phi) = \{\alpha \in \Sigma : \alpha(H_\phi) = 0\}$ for the parabolic type of ϕ. As before Z_Θ stands for the centralizer of H_ϕ and Q_ϕ the Z_Θ-block reduction. Let $\mathfrak{n}_\Theta^-$ be the nilradical of $\mathfrak{p}_\Theta^-$ (which identifies to the tangent space at the origin of $\mathbb{F}_\Theta$). The group Z_Θ normalizes $\mathfrak{n}_\Theta^-$, and hence acts linearly on $\mathfrak{n}_\Theta^-$ by the adjoint representation. Hence, we can build the associated bundle

$$\mathcal{V}_\Theta = Q_\phi \times_{Z_\Theta} \mathfrak{n}_\Theta^- \to X.$$

Since the Z_Θ-action is linear, it follows that $\mathcal{V}_\Theta \to X$ is a vector bundle and the flow Φ_t induced by ϕ_t on $\mathcal{V}_\Theta$ is linear.

Let b_Θ be the origin in $\mathbb{F}_\Theta$ and define the subset

$$\mathbb{B}_\Theta = Q_\phi \cdot N_\Theta^- b_\Theta$$

and the mapping $\Psi : \mathcal{V}_\Theta \to \mathbb{B}_\Theta$,

$$\Psi(q \cdot X) = q \cdot (\exp X) b_\Theta \qquad q \in Q_\phi, X \in \mathfrak{n}_\Theta^-.$$

Note that Ψ is well defined because if $q' \cdot X' = q \cdot X$ then there exists $g \in Z_\Theta$ with $q' = q \cdot g$ so that $X' = \mathrm{Ad}(g) X$. But $\exp X' = g \exp X g^{-1}$ and $g b_\Theta = b_\Theta$ so that $q' \cdot (\exp X') b_\Theta$ coincides with $q \cdot (\exp X) b_\Theta$. Moreover, Ψ is a homeomorphism because the mapping $X \in \mathfrak{n}_\Theta^- \mapsto (\exp X) b_\Theta \in N_\Theta^- b_\Theta$ is a homeomorphism (by Proposition 3.6 of [11]).

Proposition 5.4. *The following statements are true.*

1. *$\mathbb{B}_\Theta$ is an open and dense ϕ_t-invariant subset of $\mathbb{E}_\Theta$ which contains the attractor component $\mathcal{M}_\Theta^+ = \Psi(\mathcal{V}_\Theta^0)$, where $\mathcal{V}_\Theta^0$ is the zero section of $\mathcal{V}_\Theta$.*

2. ϕ_t and Φ_t are conjugate under Ψ:

$$\phi_t(\Psi(v)) = \Psi(\Phi_t(v)), \qquad v \in \mathcal{V}_\Theta.$$

Proof: The ϕ_t-invariance of $\mathbb{B}_\Theta$ follows from the ϕ_t-invariance of Q_ϕ. To see that it is open and dense we note that $N_\Theta^- b_\Theta$ is the open Bruhat cell in $\mathbb{F}_\Theta$, hence $q \cdot N_\Theta^- b_\Theta$ is open and dense in its fiber. Making q run through Q_ϕ, the result follows. Now,

$$\mathcal{M}_\Theta^+ = Q_\phi \cdot b_\Theta = \Psi\left(\mathcal{V}_\Theta^0\right) \subset \mathbb{B}_\Theta,$$

concluding the proof of the first statement. To see the conjugacy take $v = q \cdot X$, $q \in Q_\phi$, $X \in \mathfrak{n}_\Theta^-$. Then by definition $\Phi_t(v) = \phi_t(q) \cdot v$, so that

$$\Psi(\Phi_t(v)) = \phi_t(q) \cdot (\exp X) b_\Theta = \phi_t(q \cdot \exp X) b_\Theta = \phi_t(\Psi(v)),$$

concluding the proof. □

Finally we endow $\mathcal{V}_\Theta \to X$ with a natural metric $(\cdot, \cdot)$ given by

$$(r \cdot X, r \cdot Y) = B_\theta(X, Y), \quad r \in R_\Theta,\ X, Y \in \mathfrak{n}_\Theta^-$$

where $B_\theta(\cdot, \cdot)$ is the inner product in the Lie algebra defined by the Cartan involution θ. That this in fact defines a metric in the whole $\mathcal{V}_\Theta$ follows from the Iwasawa decomposition $Q_\phi = R_\phi AN(\Theta)$, where $AN(\Theta)$ normalizes $\mathfrak{n}_\Theta^-$.

6 $\mathcal{W}$-invariance of Lyapunov exponents

The purpose of this section is to prove the invariance of the set of $\mathfrak{a}$-Lyapunov exponents under the Weyl group (see Corollary 6.8). The proof is based on the concept of regular sequences in a symmetric space (cf. [16]).

6.1 Regular sequences in G

We discuss here some asymptotic properties of sequences in G that will be used afterwards in the proof of the invariance of the $\mathfrak{a}$-Lyapunov exponents under the Weyl group. Most of the results here are taken from [16], although we need to adapt them to our $\mathfrak{a}$-valued exponents.

For a sequence $g_k \in G$, $k \in \mathbb{Z}^+$, there are two related limits in $\mathfrak{a}$ that show up in the Iwasawa or Cartan decompositions of g_k.

1. The Lyapunov exponent of g_k at $b \in \mathbb{F}$:

$$\lambda\left(g_k, b\right) = \lim_{k \to +\infty} \frac{1}{k} \log \mathsf{a}(g_k u),$$

where $b = ub_0$, $u \in G$. This limit depends only on b because if $b = u'b_0$, $u' \in G$, then $u' = up$, for $p \in P$, so by Equation (2) one has $\mathsf{a}(g_k u') = \mathsf{a}(g_k u) + \mathsf{a}(p)$, and in the limit the term $\mathsf{a}(p)$ disappears.

2. The polar exponent of g_k:

$$\lambda^+\left(g_k\right) = \lim_{k \to +\infty} \frac{1}{k} \log \mathsf{a}^+(g_k),$$

which depends on the sequence only, contrary to the Lyapunov exponents.

Of course these limits may not exist. Next consider the left coset symmetric space $K \backslash G$ (we use this form instead of G/K to match to the Iwasawa decomposition $G = KAN$ as well as to the right action of G on the principal bundle $Q \to X$). Let d be the G-invariant distance in $K \backslash G$, which is uniquely determined by

$$d(x_0 \cdot \exp(X), x_0) = |X|_\theta, \quad X \in \mathfrak{s}$$

where x_0 is the origin of $K \backslash G$.

Following [16] a sequence $g_k \in G$ is said to be regular if there exists $D \in \mathfrak{s}$ such that $d\left(x_0 \cdot g_k, x_0 \cdot \exp kD\right)$ has sublinear growth as $k \to +\infty$, that is, if

$$\frac{1}{k} d\left(x_0 \cdot g_k, x_0 \cdot \exp kD\right) \to 0. \tag{3}$$

In this case we say that D is the asymptotic ray of g_k. (More precisely, $x_0 \cdot g_k$ is asymptotic to the geodesic ray $x_0 \cdot \exp tD$, $t > 0$.) If $u \in K$ it follows that the sequence $g_k u$ is also regular with asymptotic ray $\mathrm{Ad}\left(u^{-1}\right) D$.

Write the polar decomposition of g_k as $g_k = u_k h_k v_k \in K\left(\mathrm{cl} A^+\right) K$. Then one of the main results of [16] says that the sequence is regular if and only if it satisfies the following two conditions (see [16], Theorem 2.1):

1. The polar exponent $\lambda^+\left(g_k\right) = \lim \frac{1}{k} \log h_k$ (or equivalently the limit $h_k^{1/k}$) exists, and

2. $\frac{1}{k}d(x_0 \cdot g_k,\, x_0 \cdot g_{k+1}) \to 0$, that is, $x_0 \cdot g_k$ has sublinear growth in $K\backslash G$.

Moreover, in this case the asymptotic ray of g_k is given by $D = \mathrm{Ad}(u)H^+$ where $u \in K$ and $H^+ = \lambda^+(g_k) \in \mathrm{cl}\mathfrak{a}^+$ is the polar exponent of g_k.

The next statement gives the Lyapunov exponents of a regular sequence.

Proposition 6.1. *Suppose g_k is a regular sequence in G with asymptotic ray $D \in \mathfrak{s}$. Then the following statements hold.*

1. *Suppose $D \in \mathrm{cl}\mathfrak{a}^+$ and take $y \in P_D^-$. Then the sequence $g_k y$ is regular with the same asymptotic ray D.*

2. *Suppose $D \in \mathfrak{a}$. Then $\lim_{k\to+\infty} \frac{1}{k}\log \mathsf{a}(g_k) = D$.*

3. *Let $H^+ \in \mathrm{cl}\mathfrak{a}^+$ be the polar exponent of g_k. Then the Lyapunov exponent at $b \in \mathrm{st}(D,w)$ is given by $\lambda(g_k, b) = w^{-1}H^+$.*

Proof: For the first statement we note that since the eigenvalues of $\mathrm{ad}(D)$ on $\mathfrak{p}_D^-$ are ≤ 0, it follows from the Iwasawa decomposition of $y \in P_D^-$ that the sequence $(\exp kD)\, y\, (\exp(-kD))$ is bounded in P_D^- so that

$$d(x_0 \cdot (\exp kD)\, y,\, x_0 \cdot \exp kD) = d(x_0 \cdot (\exp kD)\, y\, (\exp(-kD)),\, x_0)$$

is bounded as a function of k (cf. [13], Proposition 3.9). But

$$d(x_0 \cdot g_k y,\, x_0 \cdot \exp kD) \leq d(x_0 \cdot g_k,\, x_0 \cdot \exp kD) + d(x_0 \cdot \exp kD,\, x_0 \cdot (\exp kD)\, y^{-1}).$$

So that $\frac{1}{k}d(x_0 \cdot g_k y,\, x_0 \cdot \exp kD) \to 0$, because g_k is asymptotic to $\exp kD$ and the last term is bounded.

For the second statement write $g_k = u_k a_k n_k \in KAN$. We have

$$|\log a_k - kD|_\theta = d(x_0 \cdot a_k, x_0 \cdot \exp kD)$$

and by a well known inequality of horospherical coordinates (cf. Corollary 1.11 of [19]) the right hand side is bounded above by

$$d(x_0 \cdot a_k n_k, x_0 \cdot \exp kD) = d(x_0 \cdot g_k, x_0 \cdot \exp kD).$$

Therefore, $\frac{1}{k}\log a_k \to D$.

Now, recall that $\mathrm{st}(D,w) = uP_{H^+}^- w b_0$ where $u \in K$ satisfies $D = \mathrm{Ad}(u)H^+$ and $b_0 \in \mathbb{F}$ is the origin. Hence, for $b \in \mathrm{st}(D,w)$ there exists $y \in P_{H^+}^-$ such

that $b = uywb_0$. In this case $\lambda(g_k, b) = \lim_{k\to+\infty} \frac{1}{k} \log a_k$ where $g_k uyw = u'_k a_k n_k \in KAN$. Since $u \in K$ the sequence $g_k u$ is regular with asymptotic ray H^+, so by the first statement the same holds for $g_k uy$. But $w \in K$ hence $g_k uyw$ is asymptotic to the ray $\mathrm{Ad}(w^{-1}) H^+ \in \mathfrak{a}$. Therefore, $\lambda(g_k, b) = \mathrm{Ad}(w^{-1}) H^+$, as follows from the second statement, concluding the proof. □

In [16] it is also given the following characterization of regularity in terms of the Lyapunov exponents $\lambda(g_k, b)$.

Proposition 6.2. *The sequence g_k is regular if and only if it has sublinear growth and the Lyapunov exponent $\lambda(g_k, b)$ exists at some (and hence at all) $b \in \mathbb{F}$.*

Proof: See [16], Theorem 2.5 and its corollary where it is proved that g_k is regular it it has sublinear growth and $\lim \frac{1}{k} \log a_k$ exists, where $g_k = u'_k a_k n_k \in KAN$ is the Iwasawa decomposition. But as mentioned above g_k is regular if and only if $g_k u$, $u \in K$, is regular. Hence the existence of $\lambda(g_k, b)$ at some $b \in \mathbb{F}$ together with sublinear growth implies regularity. □

Now we specialize the above results for sequences g_k taking values in a centralizer subgroup Z_Y, $Y \in \mathfrak{a}$. By the block reduction dicussed in Section 5 the sequences in G giving rise to our exponents will ultimately belong to a Z_Y.

Proposition 6.3. *Take $Y \in \mathrm{cl}\mathfrak{a}^+$ and put $\Theta = \{\alpha \in \Sigma : \alpha(Y) = 0\}$. Let $g_k \in Z_Y$ be a regular sequence in G. Then the asymptotic ray of g_k has the form $D = \mathrm{Ad}(u)H$ with $u \in K_\Theta$ and $H \in \mathfrak{a}$ such that $\alpha(H) \geq 0$ for all $\alpha \in \Theta$.*

Proof: Since $Z_Y = Z_\Theta$ we may take the decomposition $g_k = m_k g'_k h_k$, with $m_k \in M$, $g'_k \in G(\Theta)$, $h_k = \exp H_k \in A_\Theta$, where $H_k \in \mathfrak{a}_\Theta$. Let D be the asymptotic ray of g_k so that

$$d(x_0 \cdot g_k,\, x_0 \cdot \exp kD) = d(x_0 \cdot g'_k h_k,\, x_0 \cdot \exp kD).$$

This implies that $g'_k h_k$ is also regular in G. By Theorem 2.5 of [16] it follows that g'_k is regular in $G(\Theta)$ and the limit $\frac{1}{k} H_k \to \widehat{H}$ exists. Hence, there exists $H' \in \mathrm{cl}\mathfrak{a}(\Theta)^+$ and $u \in K(\Theta)$ such that

$$\frac{1}{k} d'(x'_0 \cdot g'_k,\, x'_0 \cdot (\exp kH')\, u^{-1}) \to 0,$$

where d' is the distance in the symmetric space $K(\Theta)\backslash G(\Theta)$ with origin $x_0' = K(\Theta)$. Since the embeeding $K(\Theta)\backslash G(\Theta) \hookrightarrow K\backslash G$ is isometric, this limit still holds with the distance d of $K\backslash G$. Now $x_0 \cdot g_k' h_k = x_0 \cdot g_k$, and since $h_k = \exp(H_k) \in A_\Theta$ comutes with $u \in K_\Theta$ and A is abelian we multiply by h_k on both arguments of the distance function in the previous limit to conclude that

$$\frac{1}{k} d(x_0 \cdot g_k,\, x_0 \cdot \exp(kH' + H_k)u^{-1}) \to 0.$$

Put $H = H' + \widehat{H}$. Then the above limits show that

$$\frac{1}{k} d(x_0 \cdot g_k,\, x_0 \cdot \exp k(H' + \widehat{H})u^{-1}) \to 0,$$

so that the asymptotic ray of g_k is $D = \mathrm{Ad}(u)H$. Here $u \in K_\Theta$ and $\alpha(H) = \alpha(H') \geq 0$ for all $\alpha \in \Theta$ as claimed. □

6.2 Lyapunov exponents

Let $Q = R \cdot AN$ be an Iwasawa decomposition of the principal bundle Q, where R is a K-reduction of Q. As in Section 4 the logarithm of the A-component gives rise to the $\mathfrak{a}$-cocycle $\mathsf{a}(t, \xi)$ over the flag bundle $\mathbb{F}Q$ and to the $\mathfrak{a}$-Lyapunov exponent of the flow ϕ_t

$$\lambda(\xi) = \lim_{t \to +\infty} \frac{1}{t} \mathsf{a}(t, \xi)$$

in the direction of $\xi \in \mathbb{F}Q$.

It is useful to write down limits at the bundle level in terms of sequences in the group level and hence to obtain results about the Lyapunov exponents of the flow from the asymptotics in G. The following statement follows directly from the definitions.

Proposition 6.4. *Let $r \in R$ and write $\phi_t(r) = r_t \cdot g_t$, with $r_t \in R$, $g_t \in G$, $t \in \mathbb{T}$. If $b_0 \in \mathbb{F}$ is the origin and $b = ub_0$, $u \in K$, then*

$$\lambda(r \cdot b) = \lambda(r \cdot ub_0) = \lim_{t \to +\infty} \frac{1}{t} \log a_t, \quad u \in G$$

where $g_t u = u_t' a_t n_t \in KAN$ is the Iwasawa decomposition.

In other words any Lyapunov exponent of the flow is a Lyapunov exponent of a sequence in G.

Let $Q = R \cdot S \simeq R \times S$ be a Cartan decomposition of the bundle (see Section 4). As before we let $\mathsf{S} : Q \to S$ stand for the projection onto $S \simeq K\backslash G$. For any initial condition $q \in Q$ we define the sequence $\mathsf{S}(\phi_k(q)) \in S$, $k \in \mathbb{Z}^+$.

In what follows we say that $q \in Q$ is a regular point for the flow in case $\mathsf{S}(\phi_k(q))$ is a regular sequence in $S \subset G$ in the sense of last subsection. We note that q is a regular point if and only if $q \cdot g$ is regular for every $g \in G$. That is, regularity depends only on the fiber of q. In this case we say that the base point $x = \pi\,(q) \in X$ is regular for the flow.

We intend to show that every $\mathfrak{a}$-Lyapunov exponent of the flow ϕ_t is a Lyapunov exponent of some regular sequence $\mathsf{S}(\phi_k(q))$. To this purpose we check first that as a consequence of continuity combined with compactness of the base space X the sublinear growth of $\mathsf{S}(\phi_k(q))$ always holds.

Proposition 6.5. *For any $q \in Q$ we have $\frac{1}{k}d(x_0 \cdot s_k,\, x_0 \cdot s_{k+1}) \to 0$ where $s_k = \mathsf{S}(\phi_k(q))$.*

Proof: Write $q_k = \phi_k(q) \in Q$ and let $q_k = r_k \cdot s_k \in R \cdot S$ be its Cartan decomposition. Clearly $q_{k+1} = \phi_1(q_k)$ so that

$$q_{k+1} = r_{k+1} \cdot s_{k+1} = \phi_1(q_k) = \phi_1(r_k) \cdot s_k,$$

by right invariance. Therefore, $s_{k+1} = \mathsf{S}(q_{k+1}) = \mathsf{S}(\phi_1(r_k) \cdot s_k) = \mathsf{S}(\mathsf{S}(\phi_1(r_k))s_k)$ (see Section 4). Hence

$$\begin{aligned} d(x_0 \cdot s_k,\, x_0 \cdot s_{k+1}) &= d(x_0 \cdot s_k,\, x_0 \cdot \mathsf{S}(\phi_1(r_k))s_k) = \\ &= d(x_0,\, x_0 \cdot \mathsf{S}(\phi_1(r_k))) = |\log \mathsf{S}(\phi_1(r_k))|. \end{aligned}$$

By compactness of the base space we have that R is compact. From the continuity of ϕ_t and of the Cartan decomposition it follows then that $d(x_0 \cdot s_k,\, x_0 \cdot s_{k+1})$ is bounded, which establishes the result. □

Now we can prove that any Lyapunov exponent of the flow on Q comes from a regular sequence in G.

Proposition 6.6. *Let $r \in R$. Then the following statements are equivalent.*

1. *r is a regular point for the flow. Denote by $D \in \mathfrak{s}$ the asymptotic ray and by $H^+ \in \mathrm{cl}\mathfrak{a}^+$ the polar exponent of $\mathsf{S}(\phi_k(r))$.*

2. *The Lyapunov exponent $\lambda(r\cdot b)$ exists in one, and hence in any direction $r \cdot b$, $b \in \mathbb{F}$, along the fiber of r. In that case $\lambda(r \cdot b) = w^{-1}H^{+}$ for any $b \in \mathrm{st}(D, w)$.*

Proof: Take the Cartan decomposition $\phi_k(r) = r_k \cdot s_k \in R \cdot S$. Then $s_k = \mathsf{S}(\phi_k(r))$ has sublinear growth. By Proposition 6.4 the Lyapunov exponent at $r\cdot b$, $b \in \mathbb{F}$, is the Lyapunov exponent at b of the sequence s_k. Now the result follows by Proposition 6.2 which ensures that a sequence with sublinear growth is regular if and only if some (and hence all) Lyapunov exponent exists. □

It follows immediately that any Lyapunov exponent of the flow is the Lyapunov exponent of a regular sequence.

Corollary 6.7. *Take $\xi = r \cdot b \in \mathbb{F}Q$ and assume that its Lyapunov exponent $\lambda(\xi)$ under ϕ_t exists. Then r is a regular point for the flow and $\lambda(\xi) = \lambda(s_k, b)$ where $s_k = \mathsf{S}(\phi_k(r))$.*

By Proposition 6.1 the set of $\mathfrak{a}$-Lyapunov exponents $\lambda(g_k, b)$, $b \in \mathbb{F}$, of a regular sequence g_k is invariant under the Weyl group $\mathcal{W}$. Therefore the above corollary has as a consequence the following symmetry of the Lyapunov spectrum.

Corollary 6.8. *Take $\xi = r \cdot b \in \mathbb{F}Q$ and assume that its Lyapunov exponent $\lambda(\xi)$ for the flow ϕ_t exists. Then for every $w \in \mathcal{W}$, $w\lambda(\xi)$ is also an $\mathfrak{a}$-Lyapunov exponent of the flow.*

In other words this last result says that the whole set of Lyapunov exponents is invariant under the Weyl group. Since the Morse exponents are convex combinations of Lyapunov exponents, it follows that the same result holds for the whole Morse spectrum of the flow. This last statement will be clarified later with the description of the Morse spectrum of each chain component.

Remark: The Iwasawa decomposition of Q gives an additive cocycle over $\mathbb{F}Q$. The polar decomposition of Q can be shown to give a subaditive $\mathfrak{a}$-cocycle over X in the following way. Put $\mathsf{a}^{+}(t,x) = \mathsf{a}^{+}(\phi_t(r))$, where $r \in R$ and $\pi(r) = x$. Let $\mu \in \mathfrak{a}^{*}$ be a weight of a finite dimensional representation of G, then

$$\langle \mu, \mathsf{a}^{+}(t+s,x)\rangle \leq \langle \mu, \mathsf{a}^{+}(t,\phi_s(x))\rangle + \langle \mu, \mathsf{a}^{+}(s,x)\rangle.$$

As usual (see [16]) an application of the subadditive ergodic theorem to this cocycle shows that almost all points (with respect to an invariant measure in X) are regular, thus yielding the Multiplicative Ergodic Theorem for the $\mathfrak{a}$-Lyapunov exponents. Since these matters are exhaustively treated in the literature we do not exploit it any further.

7 Spectrum of the attractor component and Weyl chambers

In this section we prove one of the main results of this paper. It locates the Morse spectrum of the attractor chain component $\mathcal{M}^+ \subset \mathbb{F}Q$ within an open cone in $\mathfrak{a}$. This cone is read off from the parabolic type of the flow (see Corollary 7.6). The proof is based on estimates of the linear flow on the vector bundle $\mathcal{V}_{\Theta(\phi)} \to X$ discussed at the end of Section 5.

We make use of the following statement about linear flows on vector bundles.

Proposition 7.1. *Let Φ_t be a linear flow on a vector bundle $\mathcal{V} \to X$ with norm $|\cdot|$ over a compact Hausdorff base space. Suppose that the zero section $\mathcal{V}^0$ is an attractor. Then there are positive constants $C, \mu > 0$ such that*

$$\|\Phi_t\|_x \leq Ce^{-\mu t}, \qquad t \geq 0, x \in X,$$

where $\|\cdot\|$ denotes the operator norm of $|\cdot|$.

Proof: Let U be an attracting neighborhood of $\mathcal{V}^0$ in $\mathcal{V}$. Let $v \in \mathcal{V}$ be arbitrary. Then $\varepsilon v \in U$ for sufficiently small $\varepsilon > 0$ which implies that $|\Phi_t(\varepsilon v)| = \varepsilon|\Phi_t(v)| \to 0$ when $t \to +\infty$. Hence, $|\Phi_t(v)| \to 0$, when $t \to +\infty$. The result then follows by the uniformity lemma of Fenichel [12] (see also [6], Lemma 5.2.7, and [25], Theorem 2.7). □

The constant $\mu > 0$ is called a contraction exponent of the linear flow. By compactness of the base space it is easily seen to be independent of the chosen metric in $\mathcal{V}$.

We have the following estimate of the values of the roots on the $\mathfrak{a}$-Lyapunov exponent in $\Lambda_{\mathrm{Ly}}(\mathcal{M}^+)$.

Theorem 7.2. *Let $\mu > 0$ be a contraction exponent of the linear flow Φ_t on $\mathcal{V}_{\Theta(\phi)}$ given in Proposition 5.4. Then every $\mathfrak{a}$-Lyapunov exponent $\lambda \in \Lambda_{\mathrm{Ly}}(\mathcal{M}^+)$ satisfies*

$$\alpha(\lambda) \geq \mu, \quad \alpha \in \Pi^+ \setminus \langle \Theta(\phi) \rangle. \tag{4}$$

Proof: The main point in the proof is to relate the operator norm of Φ_t with the cocycle $\mathsf{a}(t, \xi)$ over the flag bundle. Let $\Theta = \Theta(\phi)$. Take $\xi \in \mathcal{M}^+$ and write $\xi = r \cdot b_0$, $r \in R_\phi$. Then for $t \in \mathbb{T}$ we have the Iwasawa decomposition $\phi_t(r) = r_t \cdot a_t n_t \in R_\Theta A N^+(\Theta)$ and

$$\mathsf{a}(t, \xi) = \log a_t.$$

On the other hand if $v = r \cdot Y \in \mathcal{V}_\Theta$ with $Y \in \mathfrak{n}_\Theta^-$ then $|v| = |Y|_\theta$. But $\Phi_t(v) = \phi_t(r) \cdot Y$ and hence

$$|\Phi_t(v)| = |\phi_t(r) \cdot Y| = |r_t \cdot \mathrm{Ad}(a_t n_t) Y| = |\mathrm{Ad}(a_t n_t) Y|_\theta .$$

Since $N(\Theta)$ centralizes $\mathfrak{n}_\Theta^-$ we have $\mathrm{Ad}(n_t)Y = Y$ so that $|\Phi_t(v)| = |\mathrm{Ad}(a_t)Y|_\theta$. Therefore

$$\|\Phi_t\|_x = \|\mathrm{Ad}(a_t)|_{\mathfrak{n}_\Theta^-}\|_\theta.$$

Now $\mathrm{Ad}(a_t)|_{\mathfrak{n}_\Theta^-}$ is positive definite so that $\|\mathrm{Ad}(a_t)|_{\mathfrak{n}_\Theta^-}\|_\theta$ equals its greatest eigenvalue. Since the eigenvalues are $e^{-\alpha(\log a_t)}$, $\alpha \in \Pi^+ \setminus \langle \Theta \rangle$, it follows that

$$\log \|\Phi_t\|_x \geq -\alpha(\log a_t) = -\alpha(\mathsf{a}(t, \xi)), \quad \alpha \in \Pi^+ \setminus \langle \Theta \rangle.$$

By Proposition 7.1 there exists $B \in \mathbb{R}$ such that $\alpha(\mathsf{a}(t, \xi)) \geq \mu t + B$ for all roots $\alpha \in \Pi^+ \setminus \langle \Theta \rangle$ and $t \geq 0$.

Finally, if $\lambda \in \Lambda_{\mathrm{Ly}}(\mathcal{M}^+)$ then $\lambda = \lambda(\xi) = \lim_{t \to \infty} \frac{1}{t} \mathsf{a}(t, \xi)$ for some $\xi \in \mathcal{M}^+$, so that $\alpha(\lambda) \geq \mu$, concluding the proof. □

Corollary 7.3. *Take $\lambda \in \Lambda_{\mathrm{Ly}}(\mathcal{M}^+)$ and write $\Theta(\lambda) = \{\alpha \in \Pi^+ : \alpha(\lambda) = 0\}$. Then $\Theta(\lambda)$ is contained in the parabolic type $\Theta(\phi)$.*

Since the Morse spectrum $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ is the convex closure of $\Lambda_{\mathrm{Ly}}(\mathcal{M}^+)$ it follows at once that the same estimate of the above theorem holds for the Morse exponents.

Corollary 7.4. *If $\lambda \in \Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ then $\alpha(\lambda) > \mu > 0$ for every $\alpha \in \Pi^+ \setminus \langle \Theta(\phi) \rangle$.*

Now we have the following lemma on root systems, which might be well known. By the lack of a reference we present a proof of it.

Lemma 7.5. *The set* $\{H \in \mathfrak{a} : \alpha(H) > 0, \alpha \in \Pi^+ \setminus \langle\Theta\rangle\}$ *is the open convex cone* $\mathrm{int}\,(\mathcal{W}_\Theta \mathrm{cl}\mathfrak{a}^+)$. *Also, two cones* $\mathrm{int}\,(w_i\mathcal{W}_\Theta \mathrm{cl}\mathfrak{a}^+)$, $w_1, w_2 \in \mathcal{W}$, *are either equal or disjoint.*

Proof: Let $\mathcal{C}_\Theta$ be the cone of those $H \in \mathfrak{a}$ such that $\alpha(H) = 0$ if $\alpha \in \Theta$ and $\beta(H) > 0$ if $\beta \in \Pi^+ \setminus \langle\Theta\rangle$. Then $\beta \in \Pi^+ \setminus \langle\Theta\rangle$ if and only if $\beta(\mathcal{C}_\Theta) > 0$ (see Warner [30], Lemma 1.2.4.1). Also $w \in \mathcal{W}_\Theta$ if and only if $wH = H$, $H \in \mathcal{C}_\Theta$ and the chambers meeting $\mathcal{C}_\Theta$ in their closures are exactly $w\mathfrak{a}^+$, $w \in \mathcal{W}_\Theta$. Since roots do not change signs on chambers, it follows that $\alpha \in \Pi^+ \setminus \langle\Theta\rangle$ is ≥ 0 on $\mathcal{W}_\Theta \mathrm{cl}\mathfrak{a}^+$.

Conversely we check that $w\mathrm{cl}\mathfrak{a}^+$, $w \in \mathcal{W}_\Theta$, are the only closures of chambers where $\Pi^+ \setminus \langle\Theta\rangle$ is ≥ 0. In fact, take $w \notin \mathcal{W}_\Theta$ and $H \in \mathcal{C}_\Theta$. Then $wH \notin \mathcal{W}_\Theta \mathrm{cl}\mathfrak{a}^+$, because otherwise w would map H to a chamber containing H in its closure and this can happen only if w fixes H, that is only if $w \in \mathcal{W}_\Theta$. Now, the the set of roots that are > 0 on wH is $w(\Pi^+ \setminus \langle\Theta\rangle)$. By [30], Theorem 1.2.4.8 (and its proof) $w(\Pi^+ \setminus \langle\Theta\rangle)$ is not contained in $\langle\Theta\rangle \cup \Pi^+$ if $w \notin \mathcal{W}_\Theta$. Hence there exists $\gamma \in \Pi^- \setminus \langle\Theta\rangle$ such that $\gamma(wH) > 0$, that is, $(-\gamma)(wH) < 0$ so that $\Pi^+ \setminus \langle\Theta\rangle$ is not positive on $w\mathfrak{a}^+$.

So far we have $\{H \in \mathfrak{a} : \alpha(H) \geq 0, \alpha \in \Pi^+ \setminus \langle\Theta\rangle\} = \mathcal{W}_\Theta \mathrm{cl}\mathfrak{a}^+$. But to take strict inequality > 0 in the left hand side amounts to take interior in the right hand side.

To prove the last statement we apply w_2^{-1} to both cones and reduce to the case where $w_2 = 1$. Now if $\mathrm{int}\,(w_1\mathcal{W}_\Theta \mathrm{cl}\mathfrak{a}^+)$ meets $\mathrm{int}\,(\mathcal{W}_\Theta \mathrm{cl}\mathfrak{a}^+)$ then, by the first part, they meet at a regular element, say $w_1 w H = w' H'$ where $w, w' \in \mathcal{W}_\Theta$ and $H, H' \in \mathfrak{a}^+$ are regular. Since $\mathcal{W}$ acts simply on the Weyl chamber $\mathfrak{a}^+$ it follows that $H = H'$ so that $(w')^{-1} w_1 w = 1$ since it fixes a regular element. This implies that $w_1 \in W_\Theta$, so that both cones coincide.

$\square$

By the above lemma we can restate Corollary 7.4 in geometric terms as follows.

Corollary 7.6. $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ *is contained in the open convex cone* $\mathrm{int}\,(\mathcal{W}_\phi \mathrm{cl}\mathfrak{a}^+)$.

8 Morse spectra and block form

In this section we combine the previous results to get the full picture of the $\mathfrak{a}$-Morse spectra of the several chain components $\mathcal{M}(w)$ with w running through the Weyl group $\mathcal{W}$. As always we assume that the flow on the base space X is chain recurrent and let $\Theta = \Theta(\phi)$ be the parabolic type of the flow ϕ_t.

It was proved before that the whole set of $\mathfrak{a}$-Lyapunov exponents is $\mathcal{W}$-invariant (see Corollary 6.8). This result will be improved now by showing that the Lyapunov spectra of the chain components are permuted to each other by the Weyl group. To this end we recall the Z_Θ-block reduction Q_ϕ of Section 5 as well as the K_ϕ-reduction R_ϕ. If b_0 stands for the origin in $\mathbb{F}$ then by Proposition 5.3 we have

$$\mathcal{M}(w) = Q_\phi \cdot wb_0 = R_\Theta \cdot wb_0.$$

Lemma 8.1. *Let $r \in R_\Theta$ be regular. Then it has asymptotic ray $D = \mathrm{Ad}(u) H^+$ where $u \in K_\Theta$ and $H^+ \in \mathrm{cl}\mathfrak{a}^+$ is the polar exponent of r. Moreover, $\Theta(H^+) \subset \Theta$.*

Proof: Since $\phi_k(r) \in Q_\phi$ we decompose $\phi_k(r) = r_k \cdot s_k \in R_\Theta \cdot (S \cap Z_\Theta)$, then $s_k = \mathsf{S}(\phi_k(r)) \in Z_\Theta$ is regular with asymptotic ray $D \in \mathfrak{s}$. By Proposition 6.3 we have $D = \mathrm{Ad}(u) H$ where $u \in K_\Theta$ and $H \in \mathfrak{a}$ is such that $\alpha(H) \geq 0$ for all $\alpha \in \Theta$.

Noting that $\mathsf{S}(\phi_k(r \cdot u)) = u^{-1} s_k u$ is a regular sequence in G with asymptotic ray $H = \mathrm{Ad}(u^{-1}) D$ it follows that $r \cdot u \in R_\Theta$ is a regular point of the flow with asymptotic ray $H \in \mathfrak{a}$. Therefore we have the Lyapunov exponent $\lambda(ru \cdot b_0) = H$ (see Proposition 6.6 and Proposition 6.1 (2)). Since $ru \cdot b_0 \in \mathcal{M}^+$ it follows that $H \in \Lambda_{\mathrm{Ly}}(\mathcal{M}^+)$ so that, applying the estimates of last section, we conclude that $\alpha(H) > 0$ for all $\alpha \in \Pi^+ \setminus \langle \Theta \rangle$ (see Corollary 7.3). It follows that $\Theta(H) \subset \Theta$. Also, $\alpha(H) \geq 0$ for all $\alpha \in \Pi^+$ so that $H \in \mathrm{cl}\mathfrak{a}^+$. This implies that H is the polar exponent of r which is denoted by $H^+ = H$ in the statement of the result. $\square$

Proposition 8.2. *Let $r \in R_\Theta$ be regular with asymptotic ray $D \in \mathfrak{s}$ and polar exponent $H^+ \in \mathrm{cl}\mathfrak{a}^+$. Then for $x = \pi(r)$ we have*

$$\bigcup_{s \in \mathcal{W}_{\Theta_\phi}} r \cdot \mathrm{fix}(D, sw) \subset \mathcal{M}(w)_x,$$

$$\Lambda_{\mathrm{Ly}}(\mathcal{M}(w)_x) = w^{-1}\mathcal{W}_{\Theta_\phi} H^+.$$

Proof: By the previous lemma we have $D = \mathrm{Ad}\,(u)\,H^+$ with $u \in K_\Theta$, $H^+ \in \mathrm{cl}\mathfrak{a}^+$ and $\Theta\,(H^+) \subset \Theta$. Let H_ϕ be as in Theorem 5.1 so that $\Theta(H^+) \subset \Theta = \Theta(H_\phi)$. Hence for any $w \in \mathcal{W}$, $s \in \mathcal{W}_\Theta$, we have $\mathrm{fix}\,(H^+, sw) \subset \mathrm{fix}\,(H_\phi, sw) = \mathrm{fix}\,(H_\phi, w)$, so that

$$r \cdot \mathrm{fix}(D, sw) = ru \cdot \mathrm{fix}(H^+, sw) \subset ru \cdot \mathrm{fix}(H_\phi, w) = \mathcal{M}(w).$$

By Proposition 6.1 (3) the above inclusion implies that

$$w^{-1}\mathcal{W}_\Theta H^+ \subset \Lambda_{\mathrm{Ly}}(\mathcal{M}(w)_x). \tag{5}$$

Now let $\xi \in \mathcal{M}(w)_x$ and write $\xi = r \cdot lwb_0$, $l \in K_\Theta$. By Proposition 6.1 (3) we have $\lambda(\xi) = s^{-1}H^+$ where $s \in \mathcal{W}$ is such that $lwb_0 \in \mathrm{st}(D, s)$. Since $u \in K_\Theta$ we have

$$lwb_0 \in \mathrm{st}(X, s) = uP^-_{\Theta(H^+)} sb_0 \subset P^-_\Theta sb_0,$$

and since $l \in K_\Theta$ one has that $wb_0 \in P^-_\Theta sb_0$. By the Bruhat decomposition it follows that $\mathcal{W}_\Theta w = \mathcal{W}_\Theta s$ so that $s^{-1} \in w^{-1}\mathcal{W}_\Theta$. Hence

$$\lambda(\xi) = s^{-1}H^+ \in w^{-1}\mathcal{W}_\Theta H^+,$$

which implies the reverse inclusion in (5). $\square$

Now we are ready to state the permutation of the Lyapunov and Morse spectra of the Morse components under the Weyl group.

Theorem 8.3. *For every $w \in \mathcal{W}$ we have*

$$\Lambda_{\mathrm{Ly}}\,(\mathcal{M}\,(w)) = w^{-1}\Lambda_{\mathrm{Ly}}(\mathcal{M}^+) \qquad \text{and} \qquad \Lambda_{\mathrm{Mo}}\,(\mathcal{M}\,(w)) = w^{-1}\Lambda_{\mathrm{Mo}}(\mathcal{M}^+).$$

Proof: The statement for the Lyapunov spectra implies that for the Morse spectra because $\Lambda_{\mathrm{Mo}}\,(\mathcal{M}\,(w))$ is the convex closure of $\Lambda_{\mathrm{Ly}}\,(\mathcal{M}\,(w))$.

For the Lyapunov spectra we recall that any Lyapunov exponent λ is $\lambda = \lambda\,(r \cdot b) = \lambda\,(s_k, b)$ for a regular point $r \in R$ and $b \in \mathbb{F}$ where $s_k = \mathsf{S}\,(\phi_k\,(r))$ (see Proposition 6.6 and Corollary 6.7). Hence,

$$\begin{aligned}\Lambda_{\mathrm{Ly}}(\mathcal{M}(w)) &= \bigcup_{x\,\mathrm{regular}} \Lambda_{\mathrm{Ly}}(\mathcal{M}(w)_x) = \bigcup_{x\,\mathrm{reg}} w^{-1}\mathcal{W}_\Theta \lambda^+(x) = \\ &= w^{-1}\left(\bigcup_{x\,\mathrm{reg}} \mathcal{W}_\Theta \lambda^+(x)\right) = w^{-1}\Lambda_{\mathrm{Ly}}(\mathcal{M}^+),\end{aligned}$$

which proves the theorem. $\square$

Remark: The above theorem says that the several Morse spectra are homeomorphic to each other. This is in consonance with the fact that the intersection of the different chain components with the fibers are homeomorphic as well. In fact, a such intersection can be identified to the orbit $K_{H_\phi}wb_0$, which in turn identifies to the coset space K_{H_ϕ}/M.

Combining this theorem with the inclusion in chambers stated in Corollary 7.6 we get the following localization result for the Morse (and Lyapunov) spectra.

Corollary 8.4. *For each $w \in \mathcal{W}$ the Lyapunov and Morse spectra $\Lambda_{\mathrm{Ly}}(\mathcal{M}(w))$ and $\Lambda_{\mathrm{Mo}}(\mathcal{M}(w))$ are contained in the open cone*

$$\mathrm{int}\left(w^{-1}\mathcal{W}_{\Theta_{phi}}\mathrm{cl}\mathfrak{a}^{+}\right).$$

It follows that $\Lambda_{\mathrm{Mo}}(\mathcal{M}(w_1)) \cap \Lambda_{\mathrm{Mo}}(\mathcal{M}(w_2)) = \emptyset$ if $\mathcal{M}(w_1) \neq \mathcal{M}(w_2)$, that is, if $\mathcal{W}_\phi w_1 \neq \mathcal{W}_\phi w_2$.

Proof: In fact, $\mathrm{int}\left(w^{-1}\mathcal{W}_{\Theta(\phi)}\mathrm{cl}\mathfrak{a}^{+}\right) = w^{-1}\ \mathrm{int}\left(\mathcal{W}_{\Theta(\phi)}\mathrm{cl}\mathfrak{a}^{+}\right)$ and the right hand side contains $\Lambda_{\mathrm{Mo}}(\mathcal{M}^{+})$. The last statement follows from the fact that the open cones $\mathrm{int}\left(w_1^{-1}\mathcal{W}_{\Theta(\phi)}\mathrm{cl}\mathfrak{a}^{+}\right)$ and $\mathrm{int}\left(w_2^{-1}\mathcal{W}_{\Theta(\phi)}\mathrm{cl}\mathfrak{a}^{+}\right)$ are either equal or disjoint. $\square$

In particular, the equalities in Theorem 8.3 say that if $w \in \mathcal{W}_\phi$, that is, if $\mathcal{M}(w) = \mathcal{M}(1) = \mathcal{M}^{+}$, then $w\Lambda_{\mathrm{Ly}}(\mathcal{M}^{+}) = \Lambda_{\mathrm{Ly}}(\mathcal{M}^{+})$ and $w\Lambda_{\mathrm{Mo}}(\mathcal{M}^{+}) = \Lambda_{\mathrm{Mo}}(\mathcal{M}^{+})$. This invariance is easily carried over to the other spectra, by taking a conjugate of $\mathcal{W}_\phi$. Namely if $s \in w^{-1}\mathcal{W}_\phi w$ then $s\Lambda_{\mathrm{Ly}}(\mathcal{M}(w)) = \Lambda_{\mathrm{Ly}}(\mathcal{M}(w))$ and $s\Lambda_{\mathrm{Mo}}(\mathcal{M}(w)) = \Lambda_{\mathrm{Mo}}(\mathcal{M}(w))$, as follows directly from the theorem. Actually, we have the following more precise result.

Corollary 8.5. *For each $w \in \mathcal{W}$ we have $w^{-1}\mathcal{W}_\phi w = \{s \in \mathcal{W} : s\Lambda_{\mathrm{Mo}}(\mathcal{M}(w)) = \Lambda_{\mathrm{Mo}}(\mathcal{M}(w))\}$. The same statement holds with Λ_{Ly} instead of Λ_{Mo}.*

Proof: In fact, $s \in \mathcal{W}$ fixes $\Lambda_{\mathrm{Mo}}(\mathcal{M}(w))$ if and only if wsw^{-1} fixes $\Lambda_{\mathrm{Mo}}(\mathcal{M}^{+})$. By the above theorem $\Lambda_{\mathrm{Mo}}(\mathcal{M}^{+})$ is invariant under $\mathcal{W}_\phi$. Conversely, if $\sigma\Lambda_{\mathrm{Mo}}(\mathcal{M}^{+}) = \Lambda_{\mathrm{Mo}}(\mathcal{M}^{+})$ then $\Lambda_{\mathrm{Mo}}(\mathcal{M}(\sigma^{-1})) = \Lambda_{\mathrm{Mo}}(\mathcal{M}^{+})$ hence $\sigma \in \mathcal{W}_\phi$. $\square$

In view of this corollary the parabolic subgroup $\mathcal{W}_\phi$ can be recovered from the Morse spectra of ϕ (more specifically from the Morse spectrum $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$). In other words the parabolic type of ϕ is completely determined by its spectra.

Another way of expressing the relationship between the exponents and the parabolic type is by comparing $\Theta(\phi)$ with the set of simple roots annihilating some Morse exponents, as stated in the next consequence of Theorem 8.3.

Corollary 8.6. *For a Morse exponent $\lambda \in \Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ write $\Theta(\lambda) = \{\alpha \in \Sigma : \alpha(\lambda) = 0\}$. Then $\Theta(\lambda) \subset \Theta(\phi)$. Also there exists $\lambda_0 \in \Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ such that $\Theta(\lambda) \subset \Theta(\phi)$.*

Proof: The inclusion $\Theta(\lambda) \subset \Theta(\phi)$ is just a restatement of Corollary 7.4. On the other hand the $\mathcal{W}_\phi$-invariance of convex set $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ implies this set contains a point λ_0 fixed by $\mathcal{W}_\phi$. This fixed point satisfies $\Theta(\lambda_0) \supset \Theta(\phi)$, hence the equality. $\square$

In other words the block form of ϕ is given by the elements in $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ with the smallest possible degree of regularity. Hence, again the block form of ϕ can be read off from the its Morse spectrum.

As an immediate consequence we mention that the parabolic type $\Theta = \Sigma$ (or block form $H_\phi = 0$) corresponds to chain transitivity of the flow on the flag bundles. Hence we have the following criterion for chain transitivity on the flag bundles.

Proposition 8.7. *The flow ϕ_t is chain transitive in some flag bundle if and only if $0 \in \Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$. In this case ϕ_t is chain transitive (and chain recurrent) on every flag bundle.*

Example: If the base X is a point then our flow is just the iteration of a single element $g \in G$ (in the discrete time case) or the action of a one-parameter group $\exp(tX)$ in G, $X \in \mathfrak{g}$ (continuous time). For the discrete-time case generated by $g \in G$, let $g = uhn$ be its Jordan-Schur decomposition (see [14], Chapter IX), where u is elliptic, h is hyperbolic, and u unipotent. Choose a Weyl chamber $A^+$of G such that $h \in \mathrm{cl}A^+$and put $h = \exp(H^+)$, $H^+ \in \mathrm{cl}\mathfrak{a}^+$. Then it can be shown that the block form of this flow is the block form of the hyperbolic part H^+ (see In [24]). Also the finest Morse decomposition in $\mathbb{F}$ is given by the fixed point set of h, that is, $\mathcal{M}(w) = \mathrm{fix}(H^+, w)$. To compute the vector exponents of this flow we fix a Cartan

and Iwasawa decomposition of G compatible with the chamber A^+. The polar exponent of g^k is precisely H^+ (cf. [29], Theorem 2.2) . So that by our results it follows that

$$\Lambda_{\mathrm{Ly}}(\mathcal{M}(w)) = \Lambda_{\mathrm{Mo}}(\mathcal{M}(w)) = w^{-1}H^+.$$

9 Representations and vector bundles

In this final section we establish some relationships between the $\mathfrak{a}$-Lyapunov and Morse spectra and the exponents of a linear flow on a vector bundle.

Let $\mathcal{V} \to X$ be a finite dimensional real vector bundle with compact Hausdorff base space and ϕ_t a linear flow on $\mathcal{V}$, assumed to be chain transitive on the base space. The bundle $Q = B\mathcal{V} \to X$ of frames of $\mathcal{V}$ is a principal bundle with structural group $\mathrm{Gl}\,(m, \mathbb{R})$, $m = \mathrm{rank}\,\mathcal{V}$. The linear flow ϕ_t maps frames into frames, hence it lifts to a right invariant flow (also denoted by ϕ_t) on $B\mathcal{V}$.

Fix in $\mathcal{V}$ a Riemannian metric $\langle \cdot, \cdot \rangle$ and let $O\mathcal{V}$ be the $\mathrm{O}\,(n)$-subbundle of orthonormal frames of $B\mathcal{V}$. Then $O\mathcal{V} \cdot \overline{A}N$ and $O\mathcal{V} \cdot S$ are, respectively, the Iwasawa and Cartan decompositions of $B\mathcal{V}$, where $\overline{A}$ is the subgroup of diagonal matrices (with positive entries), N is the subgroup of upper triangular unipotent matrices and S the set of positive definite matrices of $\mathrm{Gl}\,(m, \mathbb{R})$.

The spectra are defined by the cocycle $\mathsf{a}\,(t, r) = \log a_t \in \overline{\mathfrak{a}}$, where $r \in O\mathcal{V}$, $\overline{\mathfrak{a}}$ is the subspace of diagonal matrices and a_t is the $\overline{A}$-component in the Iwasawa decomposition of $\phi_t(r) = r_t \cdot a_t n_t \in O\mathcal{V} \cdot \overline{A}N$. This cocycle descends to the flag bundle $\mathbb{F}\mathcal{V}$ of the flags of subspaces of $\mathcal{V}$.

Write $\overline{\mathfrak{a}} = \mathbb{R} \cdot \mathrm{id} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is the subspace of zero trace diagonal matrices. Then the component of the spectra in the direction of the scalar matrices $\mathbb{R} \cdot \mathrm{id}$ is the same for any Morse component in $\mathbb{F}\mathcal{V}$ (see Section 4.4), while the $\mathfrak{a}$-component is invariant under the Weyl group (permutation group of m letters).

Now, let $\mathbb{P}\mathcal{V}$ be the projective bundle of $\mathcal{V}$. The spectra of the flow on $\mathcal{V}$ is defined by the cocycle $\mathsf{a}_{|\cdot|}\,(t, v)$ over $\mathbb{P}\mathcal{V}$ given by

$$\mathsf{a}_{|\cdot|}\,(t, v) = \log \frac{|\phi_t v|}{|v|} \qquad v \in \mathcal{V} \setminus \{0\}.$$

To get the relation between the two cocycles take $v \in \mathcal{V}$ with $|v| = 1$ and write $v = r \cdot e_1$, $r \in O\mathcal{V}$ and $e_1 \in \mathbb{R}^m$ the first basic vector. Then $\phi_t(v) = \phi_t(r) \cdot e_1$. Take the Iwasawa decomposition $\phi_t(r) = r_t \cdot a_t n_n \in \mathrm{O}\mathcal{V} \cdot AN$. Then $|\phi_t v| = |a_t n_t e_1| = |a_t e_1|$. So that

$$\mathsf{a}_{|\cdot|}(t, v) = \lambda_1(\log a_t)$$

where $\lambda_1 : \mathfrak{a} \to \mathbb{R}$ is the first eigenvalue $\lambda_1(\mathrm{diag}\{a_1, \ldots, a_m\}) = a_1$. Hence

$$\mathsf{a}_{|\cdot|}(t, v) = \lambda_1(\mathsf{a}(t, \xi))$$

for any $\xi \in \mathbb{F}\mathcal{V}$ projecting into $[v] \in \mathbb{P}\mathcal{V}$. On the other hand a chain component on the projective bundle is the projection of chain components on the flag bundle. This implies the following expressions for the spectra of the linear flow on the vector bundle.

Proposition 9.1. *Let $\mathcal{M} \subset \mathbb{P}\mathcal{V}$ be a chain component and denote by $\Lambda^{|\cdot|}_{\mathrm{Mo}}(\mathcal{M})$ and $\Lambda^{|\cdot|}_{\mathrm{Ly}}(\mathcal{M})$ its spectra. Then $\Lambda^{|\cdot|}_{\mathrm{Mo}}(\mathcal{M}) = \lambda_1\left(\Lambda_{\mathrm{Mo}}\left(\overline{\mathcal{M}}\right)\right)$ and $\Lambda^{|\cdot|}_{\mathrm{Ly}}(\mathcal{M}) = \lambda_1\left(\Lambda_{\mathrm{Ly}}\left(\overline{\mathcal{M}}\right)\right)$, where $\overline{\mathcal{M}}$ is any chain component on $\mathbb{F}\mathcal{V}$ projecting onto $\mathcal{M}$.*

In particular if the block form of ϕ_t is a regular matrix then there are $m!$, $m = \mathrm{rank}\mathcal{V}$, chain components on the maximal flag bundle $\mathbb{F}\mathcal{V}$. Each one meets the fibers in singletons. By the previous results (see Corollary 7.6 and Theorem 8.3) each Morse spectra of a chain component $\mathcal{M}(w)$ is entirely contained in the Weyl chamber $w\mathfrak{a}^+$, and so contains only regular elements. The chain components on $\mathbb{F}\mathcal{V}$ project down to m components on the projective bundle $\mathbb{P}\mathcal{V}$, which are ordered linearly. The vector bundle Morse spectrum $\Lambda^{|\cdot|}_{\mathrm{Mo}}$ of the attractor component is just the set of highest eigenvalues of the matrices in $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$. For the other components one must apply a permutation to the elements of $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ and then take the highest eigenvector. This amounts to take the successive eigenvectors of the elements of $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$. Since the elements of $\Lambda_{\mathrm{Mo}}(\mathcal{M}^+)$ (and hence of $\Lambda_{\mathrm{Ly}}(\mathcal{M}^+)$) are regular matrices this yields the simplicity of the Lyapunov spectrum of a nominal trajectory on the base space.

We note that the above proposition can be easily extended to obtain the spectra on the Grassmann bundles $\mathrm{Gr}_k\mathcal{V}$ from the vector spectra on the flag bundle $\mathbb{F}\mathcal{V}$: One needs only to replace $\mathbb{P}\mathcal{V}$ by $\mathrm{Gr}_k\mathcal{V}$ and the functional λ_1 by the functional λ_k which gives the sum of the k-first eigenvalues $\lambda_k(\mathrm{diag}\{a_1, \ldots, a_m\}) = a_1 + \cdots + a_k$.

In the general case the idea is to start with a principal bundle $Q \to X$ with structural group G and take a linear representation $\rho : G \to \mathrm{Gl}(V)$ on a vector space V. This representation yields the associated bundle $\mathcal{V} = Q \times_\rho V \to X$ which turns out to be a vector bundle. If ϕ_t is a right invariant flow on Q then the induced flow on $\mathcal{V}$ is linear.

For instance if G is semi-simple and connected then an irreducible representation ρ_μ is given by its highest weight $\mu \in \mathfrak{a}^*$. Then there is a K invariant inner product on $\mathcal{V}$ such that the norm cocycle $\mathsf{a}_{|\cdot|}(t, v) = \log \frac{|\phi_t v|}{|v|}$ is given by

$$\mathsf{a}_{|\cdot|}(t, v) = \mu(\log a_t)$$

with a_t the A-component in the Iwasawa decomposition of $\phi_t(r)$ where $v = r \cdot Y$ for a highest weight vector $Y \in V$.

Also, for $h \in \mathrm{cl}A^+$, $\log \|\rho_\mu(h)\| = \langle \mu, \log h\rangle$ (cf. [13], Lemma 4.22). Thus the vector bundle spectra of the linear flow on $\mathcal{V}$ can be recovered from the intrinsic spectra on the principal bundle. We do not go here into the details, since it requires a discussion of the chain components on the projective bundle of $\mathcal{V}$, which is not yet available.

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